



## EXAMINATION PAPER

<b>Examination Session:</b> May/June	<b>Year:</b> 2026	<b>Exam Code:</b> MATH1081-WE01
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<b>Title:</b> Calculus I (Maths Hons)
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Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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<b>Revision:</b>	
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1. (a) Find the general solution  $y(x)$  to the ordinary differential equation

$$y'' + 4y' + 4y = \cos(2x). \quad [6]$$

- (b) Find the general solution  $y(x)$  to the ordinary differential equation

$$y'' + 4y' + 4y = e^{-2x}. \quad [4]$$

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2. Let

$$f(x) = \int_0^{x^2} \frac{1-t}{2+\cos(t)} dt,$$

- (a) Find all the critical points of  $f(x)$ . For each critical point, using the first derivative test or otherwise, determine if it is a local minimum, local maximum, or a point of inflection. [7]

- (b) Use Rolle's theorem to show that there exist at least 2 points in the interval  $(-1, 1)$  such that  $f''(x) = 0$ . [3]
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3. Let  $D$  be the finite compact region between the curves  $y = x^3 - 3x$  and  $y = x$ . Calculate

$$\iint_D (x+1) dx dy. \quad [10]$$

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4. (a) Show that  $I = e^x$  is an integrating factor for the ODE

$$(y[x^2 + 2x + 1] + 2) dx + (x^2 + 1) dy = 0. \quad [3]$$

- (b) Hence, or otherwise, solve the initial value problem

$$(y[x^2 + 2x + 1] + 2) dx + (x^2 + 1) dy = 0, \quad \text{with } y(0) = 1.$$

You may leave your answer in implicit form. [7]

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5. (a) Let  $f(x) = 1 + x$  for  $0 < x < \pi$ . State the even extension of  $f(x)$  to the interval  $(-\pi, \pi)$ , and hence calculate the half-range cosine series of  $f(x)$ . [6]

- (b) By also considering the half-range sine series of  $f(x)$ , show that for all  $x \in (0, \pi)$ ,

$$\sum_{n=1}^{\infty} \left( \frac{2 \cos((2n-1)x)}{(2n-1)^2} + \frac{(1 + (1+\pi)(-1)^{n+1}) \sin(nx)}{n} \right) = \frac{\pi}{2} + \frac{\pi^2}{4}. \quad [4]$$

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6. (a) A function  $f(x, y)$  is such that its partial derivatives are given as

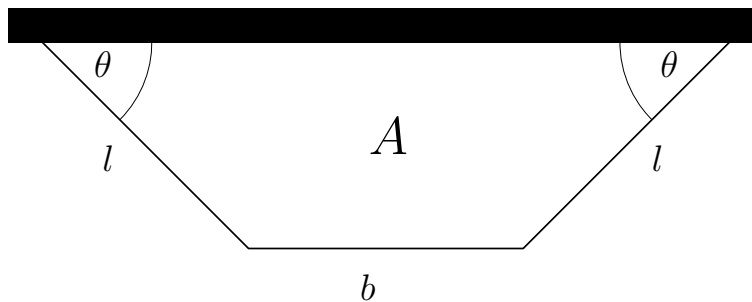
$$\begin{aligned}f_x &= -y + \cos(x + 2y), \\f_y &= -x + 2 \cos(x + 2y).\end{aligned}$$

Use the Chain Rule to calculate  $\frac{df}{dt}$  along the curve  $(x, y) = (t, t^2)$ , expressing your answer as a function of  $t$ . [2]

- (b) Find and classify the stationary points of the function

$$f(x, y) = x^2y^2 - 3x^2y + 2x^2 - y^3 + 12y. \quad [8]$$

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7. A gardener buys 9 metres of fencing in order to fence off a vegetable patch. One side of the patch lies against a wall, indicated by a thick black line in the figure below. The fencing is cut into three pieces, two of length  $l$  and a third of length  $b$ , which are to be arranged symmetrically as shown in the figure, so that the two pieces of length  $l$  make an angle  $\theta$  with the wall. The gardener wishes to maximise the area  $A$  enclosed by the fence and wall.



- (a) Find expressions for the enclosed area  $A$ , and the constraint on the total length of the fence (excluding the wall) in terms of  $l$ ,  $b$  and  $\theta$ . [3]
- (b) Use the method of Lagrange multipliers to determine the choice of  $l$ ,  $b$  and  $\theta$  which maximises the area  $A$  enclosed using the three pieces of fence, and thus find the maximum value of  $A$ . [7]
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8. A differential operator  $\mathcal{L}$  is defined as

$$\mathcal{L} = \frac{d^2}{dx^2} - 2x \frac{d}{dx}.$$

(a) Show that if  $y$  is twice differentiable then

$$\mathcal{L}y = e^{x^2} \frac{d}{dx} \left( e^{-x^2} \frac{dy}{dx} \right). \quad [2]$$

(b) Show that  $\mathcal{L}$  is a self-adjoint operator with respect to the inner product

$$(f, g) = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx,$$

where you may assume that  $f$  and  $g$  are twice differentiable and that they and their first derivatives go to zero faster than  $\exp(-x^2/2)$  as  $|x| \rightarrow \infty$ . [3]

(c) By substituting the function  $P_2(x) = x^2 + c$  into the equation  $\mathcal{L}P_2 = \lambda P_2$ , find the values of the constants  $c$  and  $\lambda$  such that  $P_2(x)$  is an eigenfunction of  $\mathcal{L}$ . [3]

(d) Explain briefly why you would expect the integral  $\int_{-\infty}^{\infty} P_2(x) \exp(-x^2)dx$  to be equal to zero. Use this to calculate  $\int_{-\infty}^{\infty} x^2 \exp(-x^2)dx$ , given that  $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$ . [2]

9. A bar of metal is arranged along the  $x$ -axis with its ends at  $x = 0$  and  $x = \pi$ . The temperature of the bar  $u(x, t)$  obeys the heat equation  $u_t = k^2 u_{xx}$ , and the ends are held at zero degrees so that  $u(0, t) = u(\pi, t) = 0$ .

(a) Show that the function

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(nx) \exp(-k^2 n^2 t)$$

solves the heat equation and satisfies the boundary conditions at the endpoints. [2]

(b) Initially the bar's temperature is given by

$$u(x, 0) = \begin{cases} x, & 0 \leq x < \pi/2, \\ \pi - x, & \pi/2 \leq x \leq \pi. \end{cases}$$

Find the solution for the temperature of the bar at later times  $u(x, t)$  in the form of the series in part a) by determining the constants  $A_n$  in this case. [4]

(c) If instead the same total heat is distributed uniformly in the bar so initially its temperature is given by

$$u(x, 0) = \pi/4$$

find the constants  $A_n$  in this case. [2]

(d) By using an approximation for the temperature distribution at late times in both the above cases, show that the ratio  $U_1(t)/U_2(t) \rightarrow 4/\pi$  as  $t \rightarrow \infty$ , where  $U_1(t)$ ,  $U_2(t)$  are the temperatures at the midpoint of the bar in the cases described in part b) and c) respectively. [2]

10. (a) State the Shift, Scaling and Derivative Theorems. [3]  
(b) Find the Fourier transform of  $\exp(-x^2/a)$  where  $a$  is a positive constant. [2]  
(c) Find the Fourier transform of  $\exp(-(x+c)^2)$  where  $c$  is a constant. [1]  
(d) Find the Fourier transform of  $(x+c)\exp(-(x+c)^2)$  where  $c$  is a constant. [2]  
(e) Find the Fourier transform of  $x\exp(-x^2-dx)$  where  $d$  is a constant. [2]

In parts b)-e) you may use that the Fourier transform of the Gaussian function  $f(x) = \exp(-x^2)$  is

$$\tilde{f}(p) = \sqrt{\pi} \exp\left(-\frac{p^2}{4}\right).$$

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