EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH1091-WE01

Title:
Linear Algebra I (Maths Hons)

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. In this question full marks cannot be obtained without showing all your working.

(a) Find the line of intersection of the two planes in $\mathbb{R}^3$ given by

$$x + 2y = 4 \qquad \text{and} \qquad x - y - z = 1\,.$$

Express your answer in both Cartesian and parameterised forms. **[5]**

(b) Find all unit vectors $\mathbf{v}$ in $\mathbb{R}^4$ which are at an angle of $2\pi/3$ to the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,.$$

[5]

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2. For what values of s does the following system of equations have a unique solution?

$$\begin{aligned} x + y - sz &= 1 \\ sx - sy + z &= 1 \\ x + (s+1)y &= 1 \end{aligned}$$

Justify your answer, quoting any results from lectures that you use. **[5]**

For those values of s for which you think the system of equations does **not** have a unique solution, determine whether they have no solutions or infinitely many solutions. When there are infinitely many solutions, find the solution set. **[5]**

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3. (a) Define what is meant by an *inverse* of a square matrix, and prove that a square matrix can have at most one inverse. **[4]**

(b) Define what is meant by a *vector subspace* of a vector space V. **[2]**

(c) Let

$$S = \left\{ p(x) \in \mathbb{R}[x]_2 \,:\, p'(1) = 0 \text{ and } \int_0^1 p(x)dx = 0 \right\}\,,$$

where $p'(1)$ denotes the value of the derivative of $p(x)$ at 1. You may assume that S is a vector subspace of $\mathbb{R}[x]_2$, the space of polynomials of degree at most 2. Find a basis for S and give a basis for $\mathbb{R}[x]_2$ that includes your basis for S, justifying your answer. **[4]**

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4. (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ to be *linearly independent*. Define also the *span* of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. **[2]**

(b) Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ are linearly independent and $\mathbf{u} \in \mathbb{R}^n$ is not an element of their span, then $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ are also linearly independent. **[4]**

(c) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are n vectors in $\mathbb{R}^n$. Let A be the $n \times n$ matrix whose columns are given by the $\mathbf{v}_i$. State and prove an 'if and only if' relation between the RREF of A and whether or not the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are a basis for $\mathbb{R}^n$. You may assume that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ being a basis of $\mathbb{R}^n$ is equivalent to them being linearly independent. **[4]**

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5. (a) Let V and W be vector spaces. Define what it means to say that a function $T\colon V \to W$ is a *linear map*, and define what is meant by the *image*, the *kernel*, the *rank* and the *nullity* of T. **[5]**

(b) You are told that the function $T\colon \mathbb{R}[x]_2 \to \mathbb{R}[x]_3$ given by

$$T(p(x)) \;=\; p'(x) + 2p(x) + \int p(x)dx$$

is a linear map. Write down its matrix with respect to the standard bases of $\mathbb{R}[x]_2$ and $\mathbb{R}[x]_3$, and compute the rank and nullity of T. **[5]**

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6. Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

and use it to find A^n for any integer $n \geq 3$. *Hint: Use the Cayley-Hamilton theorem.* **[10]**

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7. Show that the matrix

$$A = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$$

is similar to a diagonal matrix D. Find an appropriate matrix M that can convert A to D. **[10]**

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8. Let $V = \mathbb{C}^2$. For $\mathbf{z} = (z_1, z_2)$ and $\mathbf{w} = (w_1, w_2)$ in V, define

$$\langle \mathbf{z}, \mathbf{w} \rangle \;=\; z_1\overline{w_1} \;+\; a\, z_2\overline{w_2} \;+\; b\, z_1\overline{w_2} \;+\; ic\, z_2\overline{w_1},$$

where in general $a, b, c \in \mathbb{C}$. Determine all values of a, b, c for which $\langle \mathbf{z}, \mathbf{w} \rangle$ is a complex inner product on V.

For such values of a, b, c, find all vectors $\mathbf{u} \in \mathbb{C}^2$ orthogonal to the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ **[10]**

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9. Let $V = \mathbb{R}[t]_2$ denote the real vector space of all real polynomials in t of degree at most 2. Equip V with the inner product

$$\langle f, g \rangle \;=\; \int_0^1 2\, f(t)\, g(t)\, dt\,, \qquad (f, g \in V).$$

Let $U = \{\, f \in V : f(0) = 0 \text{ and } f(1) = 0 \,\}$. Find a basis for the orthogonal complement $U^{\perp}$. **[10]**

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10. Let

$$G = \{\, f_{a,b} : \mathbb{R} \to \mathbb{R} \mid f_{a,b}(x) = ax + b,\ a, b \in \mathbb{R},\ a \neq 0 \,\}.$$

Define a binary operation $\bullet$ on G by *function composition*:

$$(f_{a,b} \bullet f_{c,d})(x) = f_{a,b}(f_{c,d}(x)).$$

Prove that $(G, \bullet)$ is a group. Determine whether this group is abelian or non-abelian. **[10]**
