Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH1571-WE01-SP

Title: Single Mathematics B (2024/25 syllabus)

Time:	3 hours	
Additional Material provided:	Tables: Normal Distribution	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

1. (a) Let $A = (-1, -1, 3)$, $B = (4, 0, -3)$ and $C = (1, 2, 2)$.

(i) Find the area of the triangle in $\mathbb{R}^3$ with vertices A, B, C. **[3]**

(ii) Denoting the origin by O, find the volume of the parallelepiped with edges $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$. **[4]**

(b) A particle has its position vector in the Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ at time t given by

$$\mathbf{r}(t) = R\cos(\omega t)\mathbf{i} + 2R\sin(\omega t)\mathbf{j}$$

for some positive real constants R and ω.

(i) Compute the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of the particle. **[5]**

(ii) Express the acceleration $\mathbf{a}(t)$ in terms of the position $\mathbf{r}(t)$. **[5]**

(iii) Geometrically, describe the trajectory of the particle. **[3]**

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2. (a) Find the general solution of

$$y'' + y' - 6y = 6e^{2x}.$$

[8]

(b) Solve the following differential equation for $y = y(x)$

$$y' + 2\frac{y}{x} = x^3 y^2 ,$$

under the condition $y(1) = 1$. **[12]**

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3. Let $f(x)$ be the 2π-periodic function given on the interval $(-\pi, \pi]$ by

$$f(x) = x^2 .$$

(a) Draw the function in the interval $[-2\pi, 2\pi]$ and give the Fourier expansion of $f(x)$ in terms of trigonometric expressions. **[6]**

(b) Substituting for suitable x in the Fourier expansion of $f(x)$, prove that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} ,$$

and by a different suitable substitution deduce a value for

$$\sum_{n\geq 1} \frac{(-1)^n}{n^2} .$$

[5]

(c) State Parseval's Theorem pertaining to the Fourier series of a 2π-periodic real function. **[3]**

(d) Use Parseval's Theorem and results from the previous question parts to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4} .$$

[6]

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4. (a) Find and classify the critical points of the function

$$f(x,y) = x^3 + 3xy + y^3 - 3\,.$$ **[6]**

(b) The water temperature in a lake is given by

$$T(x,y,z) = 3x^2 - 4xy + 2y^2 - z^2 + 5,$$

where the surface is at $z = 0$. A diver at the position $(-1, 1, -3)$ begins to feel cold and wishes to warm up as quickly as possible. In which direction, written as a unit 3-vector, should the diver swim? What is the rate at which the water temperature increases in this direction? **[6]**

(c) Determine all constant parameters a and b such that the differential

$$df = (3x^2 + axy - 3y^2)\,dx + (bx^2 - 6xy + 12y)\,dy\,.$$

is exact and, for these values of a and b, determine a function $f(x,y)$ whose total differential is df. **[8]**

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5. (a) Determine all values of $a \in \mathbb{R}$ for which a function of the form

$$u(x,t) = \exp(ax + at)$$

is a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial x^2}.$$ **[6]**

(b) Consider the vector field

$$\mathbf{V} = (V_1, V_2, V_3) = \left(2xy + ze^{xz}\ln(1+y^2),\ x^2 + \frac{2ye^{xz}}{1+y^2},\ xe^{xz}\ln(1+y^2)\right).$$

Determine:

(i) $\nabla \cdot \mathbf{V}$

(ii) $\nabla \times \mathbf{V}$ **[8]**

(c) Let $\mathbf{W} = (W_1, W_2, W_3)$ be a vector field satisfying

$$\frac{\partial W_1}{\partial x} = 0\,, \quad \nabla \times \mathbf{W} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{W} = f(x),$$

where $f(x)$ is a function depending only on x (i.e. it does not depend on y and z). Show that the y-component W_2 and z-component W_3 of $\mathbf{W}$ each satisfy the two-dimensional Laplace equation; that is, show that

$$\frac{\partial^2 W_2}{\partial y^2} + \frac{\partial^2 W_2}{\partial z^2} = 0 = \frac{\partial^2 W_3}{\partial y^2} + \frac{\partial^2 W_3}{\partial z^2}\,.$$ **[6]**

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6. (a) Evaluate the integral

$$\iint_D xy\,dA$$

where D is the finite region bounded from below by the line $y = 1$ and from above by the circle $(x-1)^2 + y^2 = 5$. **[8]**

(b) Evaluate the integral

$$\frac{24}{49}\iiint_R z\,dV$$

where R is the finite region in the upper half-space (i.e. $z \geq 0$) that is bounded from above by the paraboloid $z = 10 - x^2 - y^2$ and from below by the sphere $x^2 + y^2 + z^2 = 16$. **[12]**

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7. (a) There are currently 20,000 students at Durham University, with 16000 being registered as undergraduates and 4000 as postgraduates. A survey has determined that 47% of undergraduate students engage in at least six hours of extra-curricular activities each week, while only 23% of postgraduate students do so.

(i) Show that a student selected at random engages in at least six hours of extra-curricular activities each week with probability $\frac{211}{500} = 0.422$. **[2]**

(ii) A sequence of trials are conducted where a student is selected at random and it is recorded whether that student engages in at least six hours of extra-curricular activities each week. What is the probability that in four trials at least three of the selected students engage in at least six hours of extra-curricular activities each week? **[3]**

(iii) Using an appropriate approximation, estimate the probability that in 500 trials at least 200 students who engage in at least six hours of extra-curricular activities each week are selected? **[3]**

(iv) What is the (conditional) probability that a randomly selected student is an undergraduate, given that they engage in at least six hours of extra-curricular activities each week? **[2]**

(b) Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{5(x^3-x)}{2}, & \text{if } x \in [-1,0], \\ \frac{-3(x^3-x)}{2}, & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

(i) Calculate the expectation $\mathbb{E}[X]$ and variance $\mathrm{Var}(X)$ of X. **[3]**

(ii) Let $X_1, X_2, \ldots, X_5$ be a sample of independent and identically distributed copies of X and let $\bar{X}$ be the sample mean. Calculate the expectation $\mathbb{E}[\bar{X}]$ and variance $\mathrm{Var}(\bar{X})$ of $\bar{X}$. **[3]**

(iii) Use the Central Limit Theorem to find an approximate value for the probability $\mathbb{P}(\bar{X} > 0)$ that the sample mean is greater than 0. **[4]**
