



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH2011-WE01
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Title: Complex Analysis II (2024/25 syllabus)

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Consider the function $f: \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$f(z) = f(x + iy) = (x^2 - y^2 + \sin(x)) + 2ixy.$$

Show that f is not holomorphic at any point. Justify your answer. [10]

2. Let X be a set on which we have two metrics $d_1: X \times X \rightarrow \mathbb{R}_{\geq 0}$ and $d_2: X \times X \rightarrow \mathbb{R}_{\geq 0}$. Show that the function $d: X \times X \rightarrow \mathbb{R}$ defined by

$$d(x, y) = d_1(x, y) + d_2(x, y)$$

is a metric on X . [10]

3. (a) Show that the function $v(x, y) = 3x^2y - y^3 + 4y$ is harmonic on \mathbb{C} and find a real-valued function $u(x, y)$ such that $f(x + iy) := u(x, y) + iv(x, y)$ is a holomorphic function. [7]

(b) Express $f(x + iy)$ as a function of the complex variable z , where $z = x + iy$. [3]

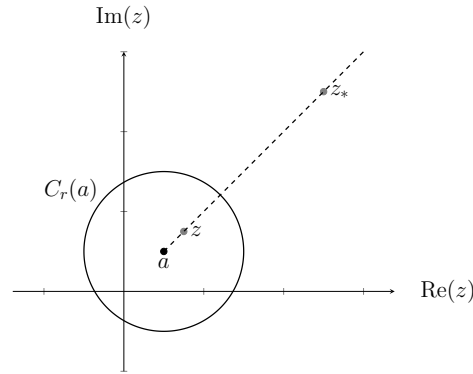
4. Find a meromorphic function f such that

$$\int_{|z|=1} f(z) dz = \int_0^{2\pi} \frac{2 \cos(\theta)}{5 - 4 \cos(\theta)} d\theta.$$

Hence, evaluate the latter integral. [10]

SECTION B

5. Let $C_r(a)$ denote the circle of radius $r > 0$ centred at $a \in \mathbb{C}$. The points $z \in \mathbb{C} \setminus \{a\}$ and $z_* \in \mathbb{C} \setminus \{a\}$ are called *symmetric* with respect to $C_r(a)$ if z and z_* lie on a ray that emerges from a (i.e. $z - a$ and $z_* - a$ have the same argument) and $|z - a| |z_* - a| = r^2$.



You may use the fact that for every $z \in \mathbb{C} \setminus \{a\}$ the requirements above define a unique point z_* such that z and z_* are symmetric with respect to $C_r(a)$.

- (a) Show that for any $r > 0$ and $a \in \mathbb{C}$ the points z and $z_* = a + \frac{r^2}{z-a}$ are symmetric with respect to $C_r(a)$ as long as $z \neq a$. [5]
- (b) Find the unique point z_* such that $z = 1$ and z_* are symmetric with respect to the circle $C_r(0)$ for a given $r > 0$. [2]
- (c) Find the unique Möbius transformation M that takes $C_{\frac{1}{2}}(0)$ to $C_1(1)$ and satisfies $M(0) = 1$ and $M(1) = 3$. [8]

Hint: You may use the following property of Möbius transformations without proof: Let M be a Möbius transformation that takes the circle C to the circle C' . If z and z_ are symmetric with respect to C then $M(z)$ and $M(z_*)$ are symmetric with respect to C' .*

6. Consider the series

$$S(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

- (a) Show that the series converges locally uniformly in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. [7]
- (b) Dirichlet's test for convergence of "alternating" complex series states that if $\{a_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of *real* number that goes to zero, and if $\{b_n\}_{n \in \mathbb{N}}$ is a sequence of complex numbers such that there exists $M > 0$ with

$$B_N = \left| \sum_{n=1}^N b_n \right| \leq M$$

for all $N \in \mathbb{N}$ then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Use Dirichlet's test to show that the series $S(z)$ converges pointwise for any point in the set $\{z \in \mathbb{C} : |z| = 1, z \neq 1\}$. [8]

Hint: You may use without proof the fact that for any $q \in \mathbb{C} \setminus \{1\}$ we have that

$$\sum_{n=1}^N q^n = \frac{q(1 - q^N)}{1 - q}.$$

7. Let $f : D \rightarrow \mathbb{C}$ be non-constant and holomorphic on a domain D containing the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose $|f(z)| = 1$ whenever $|z| = 1$.

- (a) Use the Maximum Modulus Principle to prove that $|f(z)| < 1$ for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. [5]
- (b) Show that if f had no zeros in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ then we would have

$$\frac{1}{|f(z)|} < 1, \quad \text{for } z \in \mathbb{D}.$$

Explain why this means f has at least one zero in \mathbb{D} . [3]

- (c) Let $w \in \mathbb{D}$. Show that $f(z)$ and $f(z) - w$ have the same number of zeros in \mathbb{D} , stating any results that you use. Hence, determine the image $f(\mathbb{D})$ of the unit disc under f . [7]
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8. The Fibonacci numbers F_n , for $n \geq 0$, are the non-negative integers defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2.$$

It can be shown that they satisfy the property $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}$.

Consider the power series $f(z) = \sum_{n=1}^{\infty} F_n z^n$ whose coefficients F_n are the Fibonacci numbers.

- (a) Give the radius of convergence R of f , and show that f satisfies the relation

$$f(z) = z + zf(z) + z^2f(z)$$

within its disc of convergence $B_R(0)$. [3]

- (b) Hence, find a meromorphic function g on \mathbb{C} such that f is the restriction of g to $B_R(0)$. Explain why g is the unique meromorphic function with this property and determine its poles and their orders. [5]

- (c) Calculate

$$(i) \int_{|z|=1} g(z) dz; \quad (ii) \int_{|z|=1} \frac{g'(z)}{g(z)} dz;$$

where g is the function found above. [7]