Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2031-WE01

Title:
Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $\boldsymbol{f}(\boldsymbol{x}) = \cos 2\theta\, \boldsymbol{e}_r - 2\sin(2\theta)\, \boldsymbol{e}_\theta + 2e^{2z}\, \boldsymbol{e}_z$ in cylindrical coordinates.

(a) By finding a suitable potential $g(\boldsymbol{x})$, show that $\boldsymbol{f}$ is conservative – in other words, it may be written as $\boldsymbol{f} = \boldsymbol{\nabla} g$. **[6]**

(b) Naming any theorem that you use, compute the line integral of $\boldsymbol{f}$ along an arbitrary curve from $(r_1, \theta_1, z_1) = (0, 0, 0)$ to $(r_2, \theta_2, z_2) = (1, \pi/2, 0)$. **[4]**

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2. Consider the function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\boldsymbol{f}(x, y) = \begin{pmatrix} \cos x \cos y \\ \cos x \sin y \end{pmatrix}.$$

(a) For which $(x, y) \in \mathbb{R}^2$ is $\boldsymbol{f}$ a local diffeomorphism? **[5]**

(b) Find a linear approximation to $\boldsymbol{f}$ near the point $(x, y) = (\pi/4, 0)$. **[5]**

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3. Consider the function

$$u(x, t) = \frac{1}{2t} \mathrm{e}^{-|x|/t}.$$

Show that this function defines a delta distribution in the limit $t \to 0$ (from above).

You may make use of the Hadamard Lemma:

$$h(z) = h(0) + z\, h_1(z).$$

for any bounded smooth functions $h(z)$ and $h_1(z)$. **[10]**

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4. Find the Green's function $G(x, \xi)$ for the operator

$$L = \frac{d^2}{dx^2} + \frac{d}{dx}$$

on $x \in [0, 1]$, subject to the boundary conditions

$$G(0, \xi) = 0, \quad \frac{\partial G}{\partial x}(0, \xi) = 1.$$ **[10]**

SECTION B

5. A system of curvilinear coordinates (u, v, z) is given by

$$\boldsymbol{x}(u, v, z) = a \cosh u \cos v \, \boldsymbol{e}_1 + a \sinh u \sin v \, \boldsymbol{e}_2 + z \, \boldsymbol{e}_3,$$

where $u \in [0, \infty)$, $v \in [0, 2\pi]$, $z \in \mathbb{R}$, and a is a fixed positive constant.

(a) Show that these coordinates are orthogonal. **[4]**

(b) Show that $h_u = a\sqrt{\sinh^2 u + \sin^2 v}$ and find h_v. **[3]**

(c) A closed curve is given by fixing $z = 0$ and $u = b$, for some constant $b > 0$. By evaluating a suitable surface integral parametrised with (u, v), compute the area enclosed by this closed curve. **[8]**

[*Hint:* Recall that $\cosh u = \frac{1}{2}(e^u + e^{-u})$ and $\sinh u = \frac{1}{2}(e^u - e^{-u})$.]

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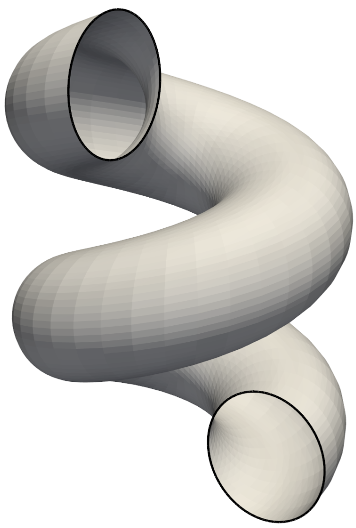
6. A surface S is described by the parametrisation

$$\boldsymbol{x}(u, v) = (3 + 2\cos u) \cos v \, \boldsymbol{e}_1 + (3 + 2\cos u) \sin v \, \boldsymbol{e}_2 + (v + 2\sin u) \, \boldsymbol{e}_3,$$

for $u \in [0, 2\pi]$ and $v \in [0, 7\pi/2]$. The surface resembles a piece of (open-ended) *spirali* pasta, as in the following picture.



(a) Compute the curl of the vector field **[3]**

$$\boldsymbol{f}(\boldsymbol{x}) = -z\boldsymbol{e}_1 + \big[z + \sin(xyz)\big]\boldsymbol{e}_2 + (x - 3)\boldsymbol{e}_3.$$

(b) By considering maximum and minimum values of the appropriate parameter (u or v), find parametrisations for each of the two circles comprising the boundary of S. Give the centre and radius of each of these circles. **[4]**

(c) Use a suitable integral theorem to find the *outward* flux of $\boldsymbol{\nabla} \times \boldsymbol{f}$ through the surface S, where $\boldsymbol{f}$ is the vector field in part (a). **[8]**

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7. (a) Let L be a linear differential operator acting on scalar functions on a bounded domain $D \subset \mathbb{R}^n$, with suitable boundary conditions imposed.
Explain how the existence of a non-trivial solution to the homogeneous adjoint problem

$$L^* w = 0$$

affects the solvability of the inhomogeneous equation

$$Lu = f.$$

Your answer should state the condition that f must satisfy. **[3]**

(b) Consider the operator

$$Lu = -\boldsymbol{\nabla} \cdot (a(\boldsymbol{x}) \boldsymbol{\nabla} u),$$

where $a(\boldsymbol{x}) > 0$ is a smooth function on a bounded domain $D \subset \mathbb{R}^n$. Assume homogeneous Dirichlet boundary conditions:

$$u = 0 \quad \text{on } \partial D.$$

Show that L is fully self-adjoint. You may use the identity

$$\boldsymbol{\nabla} \cdot (f \boldsymbol{g}) = \boldsymbol{\nabla} f \cdot \boldsymbol{g} + f \boldsymbol{\nabla} \cdot \boldsymbol{g}.$$

[8]

(c) You are told that the only solution to

$$-\boldsymbol{\nabla} \cdot (a(\boldsymbol{x}) \boldsymbol{\nabla} u) = 0, \quad u = 0 \text{ on } \partial D,$$

is the trivial solution.
Show that, for any sufficiently smooth f, the boundary value problem

$$-\boldsymbol{\nabla} \cdot (a(\boldsymbol{x}) \boldsymbol{\nabla} u) = f, \quad u = 0 \text{ on } \partial D,$$

has a unique solution. **[4]**

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8. Consider the following model for a density $u(x,t) : [0,b] \times [0,\infty) \to \mathbb{R}$:

$$\frac{\partial u}{\partial t} = Lu + h(x,t), \quad L = \frac{\partial^2}{\partial x^2} + 1,$$

subject to the boundary conditions $u(0,t) = u(b,t) = 0$. The function $h(x,t)$ is a smooth forcing function.

(a) We aim to find a general solution as an eigenfunction expansion in the spatial variable in the form:

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t) y_n(x).$$

Find the eigenfunctions $y_n(x)$ and corresponding eigenvalues λ_n. **[5]**

(b) Find the general solution to the equation with the forcing function

$$h(x,t) = \cos(t) \exp(kx).$$

You may use any results regarding eigenfunctions derived in class. **[10]**
