EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2051-WE01

Title:
Numerical Analysis II

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) Define what is meant by a floating-point number having a *finite precision* and a *finite range*. **[3]**

(b) A floating-point representation uses four *decimal* digits for the fraction part and two *decimal* digits for the exponent, allowing only normalised representation. Determine the ratio largest/smallest representable numbers. (Note: the exponent bias is irrelevant for this question.) **[2]**

(c) Let $x_0 = 1$ and $x_{k+1} = x_k/7$. If one was to compute x_k using the usual Python `float` (regarded to have smallest representable 2^{-1022} and machine epsilon 2^{-52}), what would one get for x_{200} and x_{1000}? Justify your answers. **[5]**

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2. (a) Given $f \in C^\infty$ (i.e. f is continuously differentiable as many times as needed), consider the centred difference approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \Phi(x;h).$$

Using Taylor's *theorem* (including the error term), compute the remainder Φ up to (and including) h^4. **[2]**

(b) Compute the Taylor expansion of Φ to arbitrary h^n (no need for error term). **[2]**

(c) With $R_h^{(1)} := [f(x+h) - f(x-h)]/(2h)$, derive the general Richardson extrapolation formula $R_h^{(n)}$ suitable for centred difference. **[4]**

(d) Given a small fixed h, can one keep on using Richardson iteratively to obtain arbitrarily accurate approximations of $f'(x)$? Justify your answer. **[2]**

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3. (a) Define what is meant by the 1-norm for a vector, an induced norm, and the column-sum norm for a matrix. **[4]**

(b) Considering 2×2 real matrices *only*, prove that the column-sum norm is the norm induced by the vector 1-norm. **[6]**

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4. (a) Find all values of $a \in \mathbb{R}$ such the following matrix is positive-definite:

$$A = \begin{pmatrix} a & 2 & -4 \\ 2 & 5 & 2 \\ -4 & 2 & 4 \end{pmatrix}.$$

[3]

(b) Set $a = 15$ and find a lower triangular matrix L such that $L^{\mathrm{T}} L = A$. (Note: L is not necessarily unit lower triangular.) **[7]**

SECTION B

5. (a) For $I \subset \mathbb{R}$, state what is meant by $g : I \to \mathbb{R}$ to be Lipschitz continuous. **[2]**

(b) State (do not prove) the Contraction Mapping Theorem. **[3]**

(c) Starting from the last part, state and prove the local convergence theorem for the iteration $x_{k+1} = g(x_k)$ with $g \in C^1(\mathbb{R})$. **[5]**

(d) Prove that the iteration $x_{k+1} = \sin^3 x_k$ converges for *all* $x_0 \in \mathbb{R}$. **[5]**

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6. One seeks to interpolate $f(x) = \sin^2(\pi x/6)$ with nodes $\{0, 1, 3, 5\}$.

(a) Compute a $p \in \mathcal{P}_3$ that agrees with f at those nodes. **[6]**

(b) Compute explicitly $p(2)$ and $p(4)$, as well as the errors $f - p$ at these points. **[3]**

(c) Is your p unique? Prove or give a counterexample. **[3]**

(d) Given that $|(x-0)(x-1)(x-3)(x-5)| \le 13$ for all $x \in [0,5]$, compute a bound M for $|f(x) - p(x)|$ valid for all $x \in [0,5]$. **[3]**

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7. (a) State the properties of a (real) *inner product*. **[2]**

(b) Let $\mathcal{P}_{n,0}$ be the space of polynomials of degree n with $p(0) = 0$. Show that

$$(f, g) := f'(-1)g'(-1) + f'(0)g'(0) + f'(1)g'(1)$$

defines an inner product for all f, $g \in \mathcal{P}_{3,0}$. **[5]**

(c) Let $p_n \in \mathcal{P}_{n,0}$ for $n \in \{1, 2, 3\}$ be monic polynomials orthogonal wrt the above inner product. Compute $\{p_1, p_2, p_3\}$. **[6]**

(d) Show that the polynomials you found satisfy the recurrence relation

$$p_3(x) = (x + a)p_2(x) - b p_1(x)$$

for some constants a and b. **[2]**

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8. (a) Define what is meant by a closed Newton–Cotes formula. **[2]**

(b) Show that the $n = 3$ closed Newton–Cotes formula in the interval $[a, b]$ is

$$\mathcal{I}_3(f) = \frac{b-a}{8}\Big[f(a) + 3f\Big(\frac{2a+b}{3}\Big) + 3f\Big(\frac{a+2b}{3}\Big) + f(b)\Big].$$ **[7]**

(c) One seeks to construct a quadrature formula in $[-1, 1]$ with $w(x) = 1$ that must include $x_0 = -1$. Determine the remaining nodes $\{x_1, \cdots, x_4\}$ and the coefficients $\{\rho_0, \cdots, \rho_4\}$ that give degree of exactness of (at least) seven. **[6]**
