Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2071-WE01

Title:
Mathematical Physics II

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. A particle of unit mass is confined to move on the surface of a bowl with shape $z = x^2 + y^2$.

(a) Write the kinetic energy for the particle, choosing as generalised coordinates r and θ such that

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= r^2 \end{aligned}$$

[3]

(b) Write the Lagrangian, again in terms of r and θ, assuming that the gravitational potential energy is given by gz. **[2]**

(c) Find the equations of motion following from the Lagrangian. (You do not need to solve them.) **[3]**

(d) Which coordinate is ignorable? Find the (conserved) generalised momentum associated to it. **[2]**

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2. Assume that we have a Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ that does not depend explicitly on time.

(a) Show, using the Euler-Lagrange equations, that the total energy of the system

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

is conserved. **[5]**

(b) Assume that you have a Lagrangian of the form $L = T - V$ with

$$T = \sum_{i=1}^{n} c_i(q_1, \ldots, q_n) \dot{q}_i^2$$

and c_i and V arbitrary functions of the q_i. Show that the energy is then

$$E = T + V\,.$$

[5]

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3. A one-dimensional quantum system has Hamiltonian $\hat{H}$ with orthonormal energy eigenfunctions $\psi_{E_1}(x)$ and $\psi_{E_2}(x)$, corresponding to eigenvalues E_1 and E_2. At time $t = 0$, the system is in the state

$$\psi(x, 0) = \frac{1}{\sqrt{2}} \Big(\psi_{E_1}(x) + \psi_{E_2}(x) \Big).$$

(a) Write down the wave function $\psi(x, t)$ at an arbitrary time t. **[3]**

(b) Compute the expectation value $\langle H \rangle$ and its time dependence. **[3]**

(c) Is the probability density $|\psi(x, t)|^2$ time-dependent? Motivate your answer. **[4]**

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4. Consider a particle in one dimension with wave function

$$\psi(x) = C \exp\Big(- ax^2 + ikx \Big),$$

where $a > 0$, k is real, and C is a normalisation constant.

(a) Determine the normalisation constant C. You can use

$$\int_{-\infty}^{\infty} \exp(-ax^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\,,$$

without proof. **[2]**

(b) Compute the expectation values $\langle x \rangle$ and $\langle p \rangle$. **[3]**

(c) Compute the uncertainties Δx and Δp, and determine whether this state saturates the Heisenberg uncertainty relation. **[5]**

SECTION B

5. Consider a system described by the Lagrangian

$$L_b = \sqrt{1 + \dot{x}^2 + b\dot{y}^2}\,,$$

where we assume $b > 0$.

(a) Write the generalised momenta for all ignorable coordinates in this system, and prove that they are conserved along solutions of the equations of motion. **[2]**

(b) Find all values of b for which the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

leaves the Lagrangian invariant to first order in θ, and is therefore a symmetry of the system. **[4]**

[*Hint:* $\sqrt{1+\epsilon} = 1 + \frac{1}{2}\epsilon + \ldots$]

(c) Construct the Noether charge

$$Q := \sum_i a_i \frac{\partial L}{\partial \dot{q}_i}$$

associated to the transformation in the previous part, and show, by explicitly taking its time derivative, that it is constant in time (that is, $dQ/dt = 0$) along solutions of the equations of motion precisely for the values of b you found above. **[4]**

(d) Construct the Hamiltonian H_b associated to the Lagrangian L_b, for generic values of b. **[5]**

[*Hint:* Try expressing the combination $\dot{x}^2 + b\dot{y}^2$ in terms of generalised momenta before trying to express $\dot{x}$ and $\dot{y}$ in terms of generalised momenta.]

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6. Consider a field theory for the two fields $u(x,t)$ and $w(x,t)$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 + w_t^2) - \frac{1}{2}(u_x^2 + w_x^2)$$

where $u_x := \partial u(x,t)/\partial x$, $u_t := \partial u(x,t)/\partial t$ and similarly $w_x := \partial w(x,t)/\partial x$, $w_t := \partial w(x,t)/\partial t$.

(a) Write down the equations of motion for this system. **[2]**

(b) Find the explicit form of $u(x,t)$ and $w(x,t)$ given the initial ($t = 0$) conditions:

$$u(x,0) = \sin(x) \quad ; \quad w(x,0) = 0\,,$$
$$u_t(x,0) = 0 \quad ; \quad w_t(x,0) = \cos(x)\,.$$

[6]

(c) Compute the energy-momentum tensor

$$T_{ij} := \frac{\partial \mathcal{L}}{\partial u_i} u_j + \frac{\partial \mathcal{L}}{\partial w_i} w_j - \delta_{ij} \mathcal{L}$$

explicitly for this system. (The subscripts i, j in this equation take values in $\{t, x\}$.) **[2]**

(d) Assume that the spatial direction is an interval, parameterised by the coordinate $x \in [0,1]$. Show that the total energy of the system

$$E = \int_0^1 T_{tt}\, dx$$

is conserved in time for any solution of the equations of motion, with the following boundary conditions relating the behaviour at the two ends of the interval:

$$u(0,t) = u(1,t)\,, \qquad u_x(0,t) = u_x(1,t)\,,$$

and

$$w(0,t) = w(1,t)\,, \qquad w_x(0,t) = w_x(1,t)\,.$$

[5]

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7. (a) The one-dimensional quantum harmonic oscillator with mass m and angular frequency ω has a ground state wave function

$$\psi_0(x) = Ce^{-(x/a)^2}, \qquad a = \frac{\hbar}{\sqrt{m\omega}}.$$

Using the creation operator

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega\hat{x} - i\hat{p} \right),$$

show that the spatial dependence of the first excited state must be proportional to $xe^{-(x/a)^2}$. You may ignore all normalisation constants. **[3]**

(b) Consider now a superposition of the ground and first excited states

$$\psi(x,0) = \psi_0(x) + e^{i\theta}\psi_1(x),$$

where θ is a real phase. Derive an expression for the probability current density

$$J(x) = \frac{\hbar}{2mi} \left(\bar{\psi}\, \partial_x \psi - \psi\, \partial_x \bar{\psi} \right)$$

in terms of θ. **[4]**

(c) Evaluate $J(x)$ at $x = 0$ and determine for which values of θ the current vanishes at the origin. Interpret your result physically. **[4]**

(d) Specialise your answer to the cases $\theta = \pm\frac{\pi}{2}$. Explain how the sign of $J(0)$ affects the subsequent change of the probability density for $t > 0$. **[4]**

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8. Consider a unit-size box containing two non-interacting particles of unit mass $m = 1$. The particles have coordinates x_1 and x_2 respectively, so

$$0 \leq x_1 \leq 1\,, \quad 0 \leq x_2 \leq 1\,.$$

Use units for which $\hbar = 1$.

(a) Write down the expansion of a generic wave function for the two particles in the box, on a basis of normalised energy eigen-wavefunctions for this system. Motivate your answer. **[3]**

(b) Assume that at a particular time, the system is in the state described by the wave function

$$\psi(x_1, x_2, t = t_0) = C\left[\sin(2\pi x_1)\sin(2\pi x_2) + \frac{1}{2}\sin(3\pi x_1)\sin(\pi x_2)\right]\,,$$

for some normalisation constant C. Compute this constant C.
Hint: $\int_0^1 \sin^2(n\pi x) = 1/2$. **[3]**

(c) Determine the probability density $P(x_1)$ for the system described by this wave function. Use the symmetry of $P(x_1)$ to determine the expectation value $\langle x_1 \rangle$ at $t = t_0$. **[4]**

(d) At $t = t_0$, the position of particle 2 is measured, and found to be $x_2 = 1/4$. Describe what you now know about the wave function, and give the un-normalised probability density $P(x_1)$ just after this measurement. **[3]**

(e) Does the measurement of the position of particle 2 change the expectation value $\langle x_1 \rangle$? Motivate your answer. **[2]**
