Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2647-WE01

Title: Probability II (2024/25 syllabus)

Time:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) State the defining properties of the probability function $\mathbb{P}$. **[3]**

(b) Let $X: \Omega \to \mathbb{R}$ and $Y: \Omega \to \mathbb{R}$ be random variables. What does it mean for them to be independent? **[3]**

(c) Suppose $X: \Omega \to \{0, 1, 2, \ldots\}$ is a random variable. What is the probability generating function of X? Express $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ in terms of the probability generating function of X. **[4]**

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2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Carefully state the following results.

(a) The first Borel–Cantelli Lemma for a sequence of events $(A_n)_{n\geq 1}$ in $\mathcal{F}$. **[3]**

(b) The Dominated Convergence Theorem for a sequence of real-valued random variables $(X_n)_{n\geq 1}$. **[3]**

(c) The Strong Law of Large Numbers for a sequence of real-valued random variables $(X_n)_{n\geq 1}$, assuming that $\mathbb{E}[X_n^4] \leq C$ for every n, where C is a constant. **[4]**

SECTION B

3. In the following questions, carefully justify your answers using appropriate definitions and results from the lectures. You may use any results from the lectures without proof as long as they are stated clearly.

(a) Suppose that X, Y and Z are random variables on a probability space. Show that $\max\{X, Y, Z^3\}$ is also a random variable. **[7]**

(b) Suppose that X and Y are random variables such that $\mathbb{E}[X] = \mathbb{E}[Y] \in \mathbb{R}$ and $\mathbb{P}(X \geq Y) = 1$. Show that $\mathbb{P}(X = Y) = 1$. **[8]**

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4. In the following questions, carefully justify your answers using appropriate definitions and results from the lectures. You may use any results from the lectures without proof as long as they are stated clearly.

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables on a probability space. Let

$$\phi_n(t) = \mathbb{E}\left[e^{-t|X_n|}\right] \quad \text{for } \ t \geq 0.$$

(a) Suppose that the sequence $X_n \to 0$ almost surely as $n \to \infty$. Show that $\phi_n(t) \to 1$ as $n \to \infty$, for every $t \geq 0$. **[5]**

(b) Suppose that the sequence $X_n \to 0$ in L^1 as $n \to \infty$. Show that $\sup_{0 \leq t \leq 1} |\phi_n(t) - 1| \to 0$ as $n \to \infty$. (Hint: You may want to prove the inequality $|1 - e^x| \leq |x|$ for $x \leq 0$.) **[10]**
