Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2741-WE01

Title: Methods of Mathematical Physics II

Time:	2 hours	
Additional Material provided:	Formula Sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. Consider a system described by the Lagrangian

$$L_b = \sqrt{1 + \dot{x}^2 + b\dot{y}^2}\,,$$

where we assume $b > 0$.

(a) Write the generalised momenta for all ignorable coordinates in this system, and prove that they are conserved along solutions of the equations of motion. **[2]**

(b) Find all values of b for which the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

leaves the Lagrangian invariant to first order in θ, and is therefore a symmetry of the system. **[6]**

(c) Construct the Noether charge

$$Q := \sum_i a_i \frac{\partial L}{\partial \dot{q}_i}$$

associated to the transformation in the previous part, and show, by explicitly taking its time derivative, that it is constant in time (that is, $dQ/dt = 0$) along solutions of the equations of motion precisely for the values of b you found above. **[7]**

(d) Construct the Hamiltonian H_b associated to the Lagrangian L_b, for generic values of b. **[10]**

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2. Consider a field theory for the two fields $u(x,t)$ and $w(x,t)$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 + w_t^2) - \frac{1}{2}(u_x^2 + w_x^2)$$

where $u_x := \partial u(x,t)/\partial x$, $u_t := \partial u(x,t)/\partial t$ and similarly $w_x := \partial w(x,t)/\partial x$, $w_t := \partial w(x,t)/\partial t$.

(a) Write down the equations of motion for this system. **[3]**

(b) Find the explicit form of $u(x,t)$ and $w(x,t)$ given the initial ($t=0$) conditions:

$$u(x,0) = \sin(x)\,, \qquad w(x,0) = 0\,,$$
$$u_t(x,0) = 0\,, \qquad w_t(x,0) = \cos(x)\,.$$ **[12]**

(c) Compute the energy-momentum tensor

$$T_{ij} := \frac{\partial \mathcal{L}}{\partial u_i} u_j + \frac{\partial \mathcal{L}}{\partial w_i} w_j - \delta_{ij}\mathcal{L}$$

explicitly for this system. (The subscripts i, j in this equation take values in $\{t, x\}$.) **[4]**

(d) Assume that the spatial direction is an interval, parameterised by the coordinate $x \in [0,1]$. Show that the total energy of the system

$$E = \int_0^1 T_{tt}\, dx$$

is conserved in time for any solution of the equations of motion, with the following boundary conditions relating the behaviour at the two ends of the interval:

$$u(0,t) = u(1,t)\,, \qquad u_x(0,t) = u_x(1,t)\,,$$

and

$$w(0,t) = w(1,t)\,, \qquad w_x(0,t) = w_x(1,t)\,.$$ **[6]**

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3. (a) Consider the homogeneous Maxwell equations,

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}\,,$$

and write $\mathbf{E}$ and $\mathbf{B}$ in terms of a scalar potential φ and vector potential $\mathbf{A}$ so that these equations are automatically satisfied. **[4]**

(b) Consider the sourced Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0\, \partial_t \mathbf{E}\right)\,,$$

for charge density ρ and current density $\mathbf{J}$. Write these equations in terms of φ and $\mathbf{A}$. Impose the gauge condition,

$$\nabla \cdot \mathbf{A} + \gamma\, \varepsilon_0\, \mu_0\, \partial_t \varphi = 0\,,$$

and choose γ so that the equations for φ and $\mathbf{A}$ decouple. **[8]**

(c) Assume that ρ and $\mathbf{J}$ vanish and make the ansatz,

$$\mathbf{A}(t, \mathbf{x}) = \mathbf{V}_0 \cos\left(\mathbf{k} \cdot \mathbf{x} - \omega\, t\right), \qquad \varphi(t, \mathbf{x}) = \phi_0 \cos\left(\mathbf{k} \cdot \mathbf{x} - \omega\, t\right)\,, \qquad (3.1)$$

with $\mathbf{k}$ and ω real and constant. Determine the numbers ϕ_0 and ω in terms of the vectors $\mathbf{V}_0$ and $\mathbf{k}$. **[7]**

(d) Express the Poynting vector,

$$\mathcal{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}\,,$$

for the solution (3.1), in terms of $\mathbf{k}$ and $\mathbf{V}_0$. What can you say about the direction of energy flow? **[6]**

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4. Consider an infinitely long cylinder of radius R. The cylinder carries a constant charge density ρ_0 and a constant current density $\mathbf{J}$ which is parallel to its axis. Assume that the axis of the cylinder coincides with the z-axis in cylindrical coordinates so that $\mathbf{J} = j_0\, \hat{\mathbf{e}}_z$ for positive j_0. You can assume that the charge and current densities vanish outside the cylinder.

(a) Use the integral form of Gauss' law to determine the electrostatic field $\mathbf{E}$ everywhere. **[9]**

(b) Do the same with the integral form of Ampére's law to determine the magnetostatic field $\mathbf{B}$ everywhere. **[9]**

(c) A relativistic observer $\mathcal{R}'$ is moving with velocity v along the z-axis in the positive direction. Determine the charge density ρ' and current density $\mathbf{J}'$ that $\mathcal{R}'$ measures. Write down the electric $\mathbf{E}'$ and magnetic $\mathbf{B}'$ fields, as seen by $\mathcal{R}'$. **[7]**
