Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2026 | MATH2751-WE01 |

**Title:** Probability II

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| Time: | 2 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.<br>Write your answer in the white-covered answer booklet with barcodes.<br>Begin your answer to each question on a new page. |

| Revision: | |
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ED01/2026

**1.** Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables on a probability space. Let

$$\phi_n(t) = \mathbb{E}\left[e^{-t|X_n|}\right] \quad \text{for } t \geq 0.$$

(a) Suppose that the sequence $X_n \to 0$ almost surely as $n \to \infty$. For every $t \geq 0$, show that $\phi_n(t) \to 1$ as $n \to \infty$. **[10]**

(b) Suppose that the sequence $X_n \to 0$ in $L^1$ as $n \to \infty$. Show that $\sup_{0 \leq t \leq 1} |\phi_n(t) - 1| \to 0$ as $n \to \infty$. (Hint: You may want to prove the inequality $|1 - e^x| \leq |x|$ for $x \leq 0$.) **[15]**

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**2.** (a) Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables on a probability space. Suppose that $0 < \mathbb{E}[X_n] < \infty$ for every $n$, and that

$$\sum_{n=1}^{\infty} \mathbb{P}\big(X_n \geq \mathbb{E}[X_n]\big) < \infty.$$

Prove that with probability 1,

$$\sup_{n \geq 1} \frac{X_n}{\mathbb{E}[X_n]} < \infty.$$ **[12]**

(b) Suppose $X$ is a random variable taking values in the set $\{0, 1, 2, 3, \ldots\}$, with the property that

$$\mathbb{E}\big[t^X\big] = t + \psi(t-1)$$

for every $t \in \mathbb{R}$, where the function $\psi$ has the property that it and its first three derivatives satisfy $\psi(0) = \psi'(0) = \psi''(0) = 0$ and $\psi'''(0) = 1$.

Either prove that such a random variable $X$ does not exist, or give an example of such a random variable by specifying $\mathbb{P}(X = k)$ for every integer $k \geq 0$. **[13]**

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**3.** Let $I = \{1, 2, \dots, m\}$ and let $P$ be a stochastic matrix indexed by $I$. Let $(X_n)_{n\geq 0}$ be the corresponding Markov chain, and suppose that it is recurrent. Let $N_1$ be the return time to 1, i.e., $N_1 := \inf\{n \geq 1 : X_n = 1\}$.

(a) State what it means for $(X_n)_{n\geq 0}$ to be irreducible. **[2]**

(b) Let $V_n$ denote the number of visits to state 1 up to time $n$ (not including time 0), i.e., $V_n := |\{1 \leq k \leq n : X_k = 1\}|$.

Justify that there exists a unique stationary distribution, $(\pi(i) : i \in I)$, and that as $n \to \infty$, it holds almost surely that

$$\frac{V_n}{n} \to \pi(1).$$ **[8]**

(c) Let $T_k$ be the time of the $k$th return to 1, i.e., $T_k := \inf\{n > 0 : V_n \geq k\}$. Justify that as $k \to \infty$,

$$\frac{T_k}{k} \to \mathbb{E}[N_1 | X_0 = 1].$$ **[10]**

(d) Carefully show that

$$\lim_{n\to\infty} \frac{V_n}{n} = \left(\lim_{k\to\infty} \frac{T_k}{k}\right)^{-1},$$

and thereby express $\mathbb{E}[N_1 | X_0 = 1]$ in terms of $\pi(1)$. **[5]**

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**4.** Let $\mathbb{N} = \{1, 2, \ldots\}$. Let $(X_n)_{n\geq 0}$ be the Markov chain on $\mathbb{N}$ with stochastic matrix

$$P_{ij} = \begin{cases} \frac{1}{i}, & j = i+1,\ i \text{ odd}, \\ \frac{i-1}{i}, & j = i-1,\ i \text{ odd}, \\ \frac{i-1}{i}, & j = i+1,\ i \text{ even}, \\ \frac{1}{i}, & j = i-1,\ i \text{ even}. \end{cases}$$

(a) Show that $(X_n)_{n\geq 0}$ is periodic with period 2. **[2]**

(b) Assume that $X_0 = 1$, so that $X_{2n}$ is always odd. It may be taken for granted that setting $Y_n := \frac{X_{2n}+1}{2}$ defines a new Markov chain $(Y_n)_{n\geq 0}$ on $\mathbb{N}$. Determine its stochastic matrix, $Q_{ij} = \mathbb{P}(Y_{n+1} = j | Y_n = i)$. **[5]**

(c) Now let $n_0 = 0$ and, for $k > 0$, let $n_k > 0$ be the $k$th time that $(Y_n)_{n\geq 0}$ makes a step, i.e., $Y_{n_k} \neq Y_{n_k - 1}$. We define $Z_k := Y_{n_k}$. Carefully show that $(Z_k)_{k\geq 0}$ is a Markov chain with stochastic matrix

$$R_{ij} = \mathbb{P}(Z_{k+1} = j | Z_k = i) = \begin{cases} \frac{2i-1}{4i-1}, & j = i+1,\ i > 0, \\ \frac{2i}{4i-1}, & j = i-1,\ i > 0, \\ 1, & j = 1,\ i = 0, \\ 0, & \text{otherwise}. \end{cases}$$

[Hint: consider using the one step method to calculate the probability that $(Y_n)_{n\geq 0}$ started from state $i$ visits $i+1$ before visiting $i-1$.] **[8]**

(d) By coupling with a symmetric random walk, prove that $(Z_n)_{n\geq 0}$, and hence also $(X_n)_{n\geq 0}$, is recurrent. You may use the fact that the symmetric random walk is recurrent. **[10]**

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