EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2761-WE01

Title:
Statistical Inference II

Time:	2 hours	
Additional Material provided:	Formula Sheet; Tables: Normal distribution, t-distribution, chi-squared distribution.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. An owner has a fussy cat. At any given meal, the owner offers the cat a particular food variety, which the cat eats with probability $\pi \in (0, 1)$. If the cat refuses the food, the owner offers a different variety, and continues until the cat eats the meal offered.

For each meal i, let $X_i \in \{1, 2, 3, \dots\}$ denote the number of food varieties offered until the cat eats. Assume that, conditional on π,

$$\mathbb{P}(X_i = x \mid \pi) = (1-\pi)^{x-1}\pi, \qquad x = 1, 2, \dots,$$

and that $X_1, \dots, X_n$ are independent given π.

A Beta prior distribution is assumed for π: $\pi \sim \text{Beta}(a, b)$.

(a) Derive the posterior distribution of π given data $x = (x_1, \dots, x_n)^T$. Express your answer in terms of n and a summary statistic of the data, and evaluate whether the Beta prior is conjugate for this likelihood. **[7]**

(b) The owner's prior beliefs satisfy $\mathbb{E}[\pi] = 0.25$ and $\mathbb{V}\text{ar}(\pi) = 0.0375$. Find values of a and b consistent with these beliefs. **[6]**

(c) Let X^* denote the number of food varieties required on the next meal. Suppose that a total of 18 food varieties were offered over $n = 8$ meals.

Using the posterior distribution from part (a), derive the posterior predictive probability mass function of $X^* \mid x$ and hence evaluate the probability that $X^* \leq 2$. **[12]**

---

2. Let $Y_1, \dots, Y_n$ be i.i.d. with *Beta prime*(a, b) density

$$f(y \mid a, b) = \frac{1}{\mathrm{B}(a,b)}\, y^{a-1}(1+y)^{-(a+b)}, \qquad y > 0,$$

where $a > 0$ and $b > 0$ and $\mathrm{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

(a) Show that this model belongs to a two-parameter exponential family.

Hence, or otherwise, identify the natural parameter vector, ϕ, and a sufficient statistic for (a, b) based on data $y = (y_1, \dots, y_n)^T$, stating its dimension. **[8]**

(b) A Bayesian analysis is to be carried out using a prior distribution for (a, b) that is conjugate to the likelihood.

Find this prior as a joint density for (a, b), up to proportionality, and simplify it as far as possible. **[9]**

(c) Derive the posterior distribution of the ϕ (and hence of (a, b)) and show that it is in the same conjugate family. You should clearly identify the posterior parameters, and your final answer should be expressed in terms of n and the sufficient statistic identified in part (a). **[8]**

---

3. An IT support service records the time (in minutes) taken to respond to each client's request. Let $Y_1, \ldots, Y_n$ denote the service times for n requests, which are assumed to be independent and identically exponentially distributed with density

$$f(y \mid \theta) = \theta e^{-\theta y}, \qquad y > 0,$$

where $\theta > 0$ is the service rate.

(a) Consider the hypotheses

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta = \theta_1,$$

where $\theta_1 > \theta_0$ are fixed and known constants.

Derive the most powerful test of H_0 against H_1, and show that the test rejects H_0 for sufficiently small values of $S = \sum_{i=1}^n Y_i$. Clearly state any results used to justify the optimality of the test. **[8]**

(b) Show that a level-α likelihood ratio test can be written as a rejection rule of the form $\bar{Y} < c$, and determine c in terms of n, α and θ_0.

Records show that the past $n = 25$ service requests had a mean service time of $\bar{y} = 870$. Assume $\theta_0 = 1/1000$ and carry out the test at the 5% level.

You may use the following results without proof: If $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Exp}(a)$, then $S = \sum_{i=1}^n Y_i \sim \text{Gamma}(n, a)$, and then $2aS \sim \chi^2_{2n}$. **[8]**

(c) Consider now the hypotheses:

$$H_0 : \mathbb{E}[Y] = \mu_0 \qquad \text{versus} \qquad H_1 : \mathbb{E}[Y] \neq \mu_0,$$

For a large sample, derive the generalised likelihood ratio test for these hypotheses, expressing the test statistic in terms of the sample mean $\bar{Y}$, and state the corresponding rejection rule. **[9]**

---

4. A manufacturer is developing a new anti-scratch screen coating for mobile phones. Let $Y_1, \ldots, Y_n$ denote independent measurements of the time (in years) until the first visible scratch appears on a coated screen under standardised wear conditions.

The manufacturer models these wear times $Y_1, \ldots, Y_n$ as i.i.d. with density

$$f(y \mid \beta, \lambda) = \frac{\beta}{\lambda} y^{\beta-1} \exp\left(-\frac{y^\beta}{\lambda}\right), \qquad y > 0,$$

where $\beta > 0$ and $\lambda > 0$.

(a) Derive the log-likelihood $\mathcal{L}(\beta, \lambda)$ based on a random sample $y = (y_1, \ldots, y_n)^T$, and show that for fixed β the maximiser over $\lambda > 0$ is

$$\hat{\lambda}(\beta) = \frac{1}{n} \sum_{i=1}^{n} y_i^\beta.$$

[5]

(b) For the remainder of this question, assume that $\beta = 2$ is fixed and known. With $\beta = 2$ known, state the MLE $\hat{\lambda}$ of λ and find the observed information $I(\hat{\lambda})$. Hence obtain an approximate 95% Wald confidence interval for λ, expressed in terms of $S = \sum_{i=1}^{n} y_i^2$. Simplify your results as far as possible. **[8]**

(c) Of particular interest to the engineer is the reliability function, i.e. the probability that the screen is scratch free beyond time $t > 0$, defined by

$$R(t) = \mathbb{P}(Y > t).$$

For $\beta = 2$, find an expression for $R(t)$ as a function of t, and use the delta method to obtain an expression for an approximate 95% confidence interval for $R(t_0)$ at a fixed time $t_0 = 3$. **[12]**
