Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2781-WE01

Title:
Algebra II

Time:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. (a) Show that $1 + 3\sqrt{-5}$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$. **[7]**

(b) Write $x^3 + \bar{6} \in (\mathbb{Z}/7)[x]$ as a product of irreducible polynomials over $\mathbb{Z}/7$. **[7]**

(c) Let F be a field. Prove that if $f(x) \in F[x]$ is such that $(f(x))$ is a maximal ideal of $F[x]$, then $f(x)$ is irreducible in $F[x]$. **[11]**

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2. Let R be the ring $(\mathbb{Z}/3)[x]/(x^2 + \bar{1})$.

(a) Prove that R is isomorphic to $\mathbb{Z}[x]/(3, x^2 + 1)$. **[8]**

(b) Find all the elements of R. You must show that your list contains all the elements and that the elements in your list are distinct. **[8]**

(c) Show that R is a field and find the multiplicative inverse of the element $x^7 + \bar{2}x + \bar{1} + (x^2 + \bar{1})$. **[9]**

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3. Let G be a group. The *commutator subgroup* G' is the subgroup of G generated by all elements of the form $xyx^{-1}y^{-1}$, $x, y \in G$.

(a) Show that $G' = \{e\}$ if and only if G is abelian. **[5]**

(b) Show that $S_3' = A_3$. Deduce from this that $S_n' = A_n$ for every $n \geq 3$. **[5]**

(c) Show that for any surjective homomorphism $\varphi : G \to H$ one has $\varphi(G') = H'$. **[5]**

(d) Deduce from (c) that G' is normal in G. **[5]**

(e) Show that G/G' is abelian. Moreover, let N be a normal subgroup of G such that G/N is abelian. Show that $G' \subseteq N$. (In other words, G' is the *minimal* normal subgroup such that the quotient is abelian.) **[5]**

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4. (a) Find all abelian groups (up to isomorphism) of order 80. **[5]**

(b) Find the rank and torsion coefficients of the abelian group generated by elements a, b with relation $a^2b^{-4} = e$. **[5]**

(c) Find a generator of the group $(\mathbb{Z}/23)^\times$, the multiplicative group of units of $\mathbb{Z}/23$. (You may assume the group is cyclic). **[5]**

(d) Show that $(\mathbb{Z}/16)^\times$ is not cyclic. **[5]**

(e) Show that $(\mathbb{Z}/16)^\times$ can be generated by two elements. **[5]**
