EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2791-WE01

Title:
Complex Analysis II

Time:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1.

(a) Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = f(x+iy) = \left(x^2 - y^2 + \sin(x)\right) + 2ixy.$$

Show that f is not holomorphic at any point. Justify your answer. [**8**]

(b) Let f be a holomorphic function in a domain D. Assume that

$$\operatorname{Re}(f(z)) = (\operatorname{Im}(f(z)))^3 + \operatorname{Im}(f(z))$$

for all $z \in D$. Show that f is constant in D. [**9**]

Hint: If we write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ then the given identity can be written as

$$u(x,y) = v(x,y)^3 + v(x,y).$$

(c) Let f be a holomorphic function in a domain D. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable function that satisfies $g' > 0$. Assume that

$$\operatorname{Im}(f(z)) = g(\operatorname{Re}(f(z)))$$

for all $z \in D$. Show that f is constant in D. [**8**]

Hint: If we write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ then the given identity can be written as

$$v(x,y) = g(u(x,y)).$$

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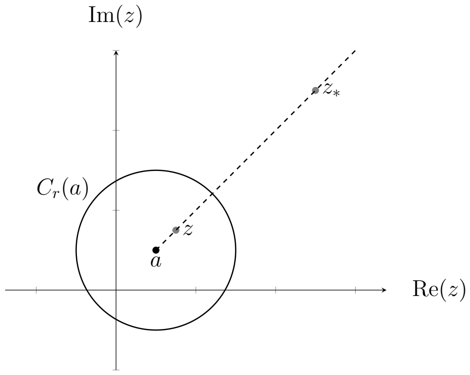
2. Let $C_r(a)$ denote the circle of radius $r > 0$ centred at $a \in \mathbb{C}$. The points $z \in \mathbb{C} \setminus \{a\}$ and $z_* \in \mathbb{C} \setminus \{a\}$ are called *symmetric* with respect to $C_r(a)$ if z and z_* lie on a ray that emerges from a (i.e. $z - a$ and $z_* - a$ have the same argument) and $|z - a|\,|z_* - a| = r^2$.



You may use the fact that for every $z \in \mathbb{C} \setminus \{a\}$ the requirements above define a unique point z_* such that z and z_* are symmetric with respect to $C_r(a)$.

(a) Show that for any $r > 0$ and $a \in \mathbb{C}$ the points z and $z_* = a + \frac{r^2}{\overline{z-a}}$ are symmetric with respect to $C_r(a)$ as long as $z \neq a$. **[10]**

(b) Find the unique point z_* such that $z = 1$ and z_* are symmetric with respect to the circle $C_r(0)$ for a given $r > 0$. **[3]**

(c) Find the unique Möbius transformation M that takes $C_{\frac{1}{2}}(0)$ to $C_1(1)$ and satisfies $M(0) = 1$ and $M(1) = 3$. **[12]**

Hint: You may use the following property of Möbius transformations without proof: Let M be a Möbius transformation that takes the circle C to the circle C'. If z and z_* are symmetric with respect to C then $M(z)$ and $M(z_*)$ are symmetric with respect to C'.

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3. Let $f : D \to \mathbb{C}$ be non-constant and holomorphic on a domain D containing the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose $|f(z)| = 1$ whenever $|z| = 1$.

(a) Use the Maximum Modulus Principle to prove that $|f(z)| < 1$ for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. **[8]**

(b) Show that if f had no zeros in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ then we would have

$$\frac{1}{|f(z)|} < 1, \qquad \text{for } z \in \mathbb{D}.$$

Explain why this means f has at least one zero in $\mathbb{D}$. **[5]**

(c) Let $w \in \mathbb{D}$. Show that $f(z)$ and $f(z) - w$ have the same number of zeros in $\mathbb{D}$, stating any results that you use. Hence, determine the image $f(\mathbb{D})$ of the unit disc under f. **[12]**

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4. The Fibonacci numbers F_n, for $n \geq 0$, are the non-negative integers defined by the recurrence relation

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad \text{for } n \geq 2.$$

It can be shown that they satisfy the property $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}$.

Consider the power series $f(z) = \sum_{n=1}^{\infty} F_n \, z^n$ whose coefficients F_n are the Fibonacci numbers.

(a) Give the radius of convergence R of f, and show that f satisfies the relation

$$f(z) = z + zf(z) + z^2 f(z)$$

within its disc of convergence $B_R(0)$. **[6]**

(b) Hence, find a meromorphic function g on $\mathbb{C}$ such that f is the restriction of g to $B_R(0)$. Explain why g is the unique meromorphic function with this property and determine its poles and their orders. **[8]**

(c) Calculate

$$(i) \int_{|z|=1} g(z)\, dz\,; \qquad\qquad (ii) \int_{|z|=1} \frac{g'(z)}{g(z)}\, dz\,;$$

where g is the function found above. **[11]**
