Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH2811-WE01

Title:
Mathematical Methods II

Time:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. A system of curvilinear coordinates (u, v, z) is given by

$$\boldsymbol{x}(u, v, z) = a \cosh u \cos v \, \boldsymbol{e}_1 + a \sinh u \sin v \, \boldsymbol{e}_2 + z \, \boldsymbol{e}_3,$$

where $u \in [0, \infty)$, $v \in [0, 2\pi]$, $z \in \mathbb{R}$, and a is a fixed positive constant.

(a) Show that these coordinates are orthogonal. **[5]**

(b) Show that $h_u = a\sqrt{\sinh^2 u + \sin^2 v}$ and find h_v. **[5]**

(c) Verify that the curve given by fixing $z = 0$ and $u = b$ (for some $b > 0$) is a closed curve. **[5]**

(d) Compute the area enclosed by the curve in part (c) by evaluating a suitable surface integral parametrised with u, v. **[10]**

[*Hint:* Recall that $\cosh u = \frac{1}{2}(e^u + e^{-u})$ and $\sinh u = \frac{1}{2}(e^u - e^{-u})$.]

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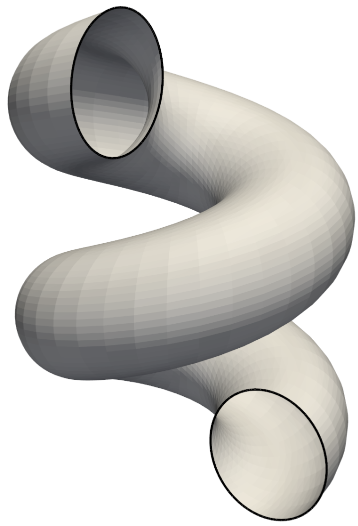
2. A surface S is described by the parametrisation

$$\boldsymbol{x}(u, v) = (3 + 2\cos u)\cos v \, \boldsymbol{e}_1 + (3 + 2\cos u)\sin v \, \boldsymbol{e}_2 + (v + 2\sin u) \, \boldsymbol{e}_3,$$

for $u \in [0, 2\pi]$ and $v \in [0, 7\pi/2]$. The surface resembles a piece of (open-ended) *spirali* pasta, as in the following picture.



(a) Compute the curl of the vector field

$$\boldsymbol{f}(\boldsymbol{x}) = -z\boldsymbol{e}_1 + \big[z + \sin(xyz)\big]\boldsymbol{e}_2 + (x - 3)\boldsymbol{e}_3.$$ **[5]**

(b) By considering maximum and minimum values of the appropriate parameter (u or v), find parametrisations for each of the two circles comprising the boundary of S. Give the centre and radius of each of these circles. **[8]**

(c) Use a suitable integral theorem to find the *outward* flux of $\boldsymbol{\nabla} \times \boldsymbol{f}$ through the surface S, where $\boldsymbol{f}$ is the vector field in part (a). **[12]**

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3. Consider the following partial differential equation for a scalar density $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + k^2 u + h(x,t),$$

where $x \in [0,b], t > 0, k > 0$ and $h(x,t)$ is a smooth forcing function. We consider solutions of this equation subject to homogeneous Neumann boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=b} = 0.$$

The solution belongs to the function space of square integrable functions whose inner product $\langle\,,\,\rangle$ is

$$\langle u, v\rangle = \int_0^b u(x)v(x)\mathrm{d}x,$$

for functions $u(x), v(x)$ belonging to this space.

(a) Consider first an equilibrium solution $u(x,t) \equiv u(x)$ on the assumption that the forcing term also only depends on x, *i.e.* $h(x,t) \equiv h(x)$. This equilibrium takes the form

$$\frac{\partial^2 u}{\partial x^2} + k^2 u = -h(x).$$

This is in the form of an inhomogeneous boundary value problem

$$Lu = f$$

encountered in class, with L a linear differential operator.
Assume an eigen-expansion solution:

$$u(x) = \sum_{n=0}^{\infty} c_n y_n(x), \text{ where } Ly_n = -\lambda_n y_n.$$

Show that the coefficients c_n, for non-zero eigenvalues take the form

$$c_n = \frac{\langle h, y_n\rangle}{\lambda_n \langle y_n, y_n\rangle}.$$

You may use any results derived in class regarding orthogonality of eigenfunctions. **[6]**

(b) Find the eigenfunctions y_n for the boundary value problem in (a). **[7]**

(c) For what values of k can we expect both, i) non-unique solutions to the problem in part (a), and ii) that the non-unique part of the solution varies spatially? **[6]**

(d) We now consider the full time-dependent problem. Assume a solution in the form:

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t) y_n(x), \text{ where } Ly_n = -\lambda y_n$$

and a forcing function

$$h(x,t) = \sin(t)\cos(\pi x/b).$$

Find the general solution for the functions $c_n(t)$. (You may assume k is such that $\lambda_n \neq 1$ and $\lambda_n \neq 0$ for any n.) **[6]**

4. (a) Consider the following function $\phi(x)$ defined on $x \in \mathbb{R}$:

$$\phi(x) = \begin{cases} \frac{1}{t}\mathrm{e}^{-x/t} & x \geq 0, \\ 0 & x < 0 \end{cases} \quad \text{where } t > 0 \text{ is a fixed constant.}$$

Show that this function defines a delta distribution in the limit $t \to 0$. The delta identity it should satisfy in this limit is

$$\int_{-\infty}^{\infty} \delta(x)h(x)\mathrm{d}x = h(0),$$

for any smooth bounded function $h(x)$.

You may use the Hadamard lemma, which states that there is some smooth function $h_1(z)$ such that we can write $h(z) = h(0) + zh_1(z)$. **[7]**

(b) Consider the following Green's function defined on $x \in [0, 1]$,

$$G(x,\xi) = \begin{cases} \dfrac{\left(e^{\xi} - e^{2-\xi}\right)\sinh(x)}{e^2 - 1}, & 0 < x < \xi, \\ \dfrac{\left(e^{x} - e^{2-x}\right)\sinh(\xi)}{e^2 - 1}. & \xi < x < 1. \end{cases}$$

Show that the function $u(x)$ defined by

$$u(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi,$$

satisfies the equation

$$\frac{d^2u}{dx^2} - u = f(x),$$

where $f(x)$ is an arbitrary continuous function.

You should use the Leibniz rule for differentiating an integral:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,\xi)d\xi = \int_{a(x)}^{b(x)} \frac{\partial g(x,\xi)}{\partial x}d\xi + \frac{db(x)}{dx}g(x,b(x)) - \frac{da(x)}{dx}g(x,a(x)),$$

and the identities $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, $\quad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$. **[8]**

(c) Use the result in part (b) to find the solution to the following boundary value problem:

$$\frac{d^2u}{dx^2} - u = f(x), \quad u(0) = 0, \quad u(1) = 1.$$ **[6]**

(d) Solve the following boundary value problem:

$$2w\frac{d^2w}{dx^2} + 2\left(\frac{dw}{dx}\right)^2 - w^2 = f(x), \quad w(0) = w(1) = 0.$$

[*Hint:* This should be possible in just two lines!] **[4]**
