EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3011-WE01

Title:
Analysis III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) State what it means for a set S to be countable. **[2]**

(b) Prove that $\mathbb{Z}$ (the set of integers) is countable.
Remember to state any results from lectures that you use. **[7]**

(c) Give an example of an uncountable set.
Note that you do **not** need to prove that the example you give is uncountable. **[1]**

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2. Let μ^* denote the Lebesgue outer measure. Let $C \subset [0,1]$ denote the middle third Cantor set.

(a) Prove that $\mu^*(C) = 0$.
Remember to state all the properties of μ^* that you use.
[You may use the fact that for any $n \in \mathbb{N}$, we can cover C by 2^n intervals of length 3^{-n}]. **[5]**

(b) Let $A = [0,1] \setminus C$. What is the Lebesgue outer measure of A?
Give a full justification of your response stating all the properties of μ^* that you use. **[5]**

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3. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and $1 \leq p < \infty$.

(a) State the definition of $L^p(E)$ and its norm $|| \cdot ||_{L^p}$. **[4]**

(b) Let $f : E \to \mathbb{R}$ be Lebesgue measurable. Prove that $\int_E |f|^p = 0$ if and only if f vanishes almost everywhere in E.
Note that it is **not** sufficient to state that the L^p-norm is a norm. **[6]**

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4. (a) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let p, q be conjugate exponents. State Hölder's Inequality for $f \in L^p(E)$, $g \in L^q(E)$. **[1]**

(b) Prove or disprove that $L^2([0,1]) \subset L^1([0,1])$. **[3]**

(c) Define $T : L^2[0,1] \to \mathbb{R}$ by

$$T(F) := \int_{[0,\frac{1}{2}]} F.$$

Prove that T is a bounded linear functional. **[6]**

SECTION B

5. (a) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = n \cdot \mathbb{1}_{(0,1/n)}(x) = \begin{cases} n, & \text{if } x \in (0, 1/n), \\ 0, & \text{otherwise,} \end{cases}$$

and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$. Then $(f_n)_{n=1}^{\infty}$ converges pointwise to f as $n \to \infty$. Does the Monotone Convergence Theorem apply to $(f_n)_{n=1}^{\infty}$? Does Fatou's Lemma apply to $(f_n)_{n=1}^{\infty}$? Justify your responses and compute the integrals $\int f_n$, $\int f$. [**7**]

(b) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let $(g_n)_{n=1}^{\infty}$ be an increasing sequence of extended real-valued nonnegative Lebesgue measurable functions on E. Suppose that the sequence of real numbers $(\int_E g_n)_{n=1}^{\infty}$ is bounded, i.e., there exists $M > 0$ such that $|\int_E g_n| \leq M$ for all $n \in \mathbb{N}$.

Prove that there exists a Lebesgue measurable function g such that the sequence $(g_n)_{n=1}^{\infty}$ converges pointwise on E to g and hence deduce that

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty.$$

In addition, prove that g is finite almost everywhere on E.

Remember to state the names of any results from lectures that you use. [**8**]

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6. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set.

(a) (i) State what it means for an extended real-valued function $h : E \to \mathbb{R} \cup \{-\infty, \infty\}$ to be Lebesgue measurable. **[2]**

(ii) Let $h : E \to \mathbb{R} \cup \{-\infty, \infty\}$ be an extended real-valued Lebesgue measurable function. For $n \in \mathbb{N}$, let $h_n : E \to \mathbb{R} \cup \{-\infty, \infty\}$ be defined by

$$h_n(x) = \begin{cases} h(x), & \text{if } |h(x)| \leq n, \\ n, & \text{if } h(x) > n, \\ -n, & \text{if } h(x) < -n. \end{cases}$$

Prove that h_n is Lebesgue measurable for all $n \in \mathbb{N}$. **[5]**

(b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of integrable functions on E that converges pointwise almost everywhere on E to f. Suppose that there is a sequence $(g_n)_{n=1}^{\infty}$ of nonnegative integrable functions on E that converges pointwise almost everywhere on E to g and satisfies

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty,$$

and

$$|f_n| \leq g_n \text{ on } E \text{ for all } n \in \mathbb{N}.$$

Prove that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

[8]

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7. (a) Let $(X, \|\cdot\|)$ be a normed linear space. State what it means for $(X, \|\cdot\|)$ to be a Banach space. **[2]**

(b) Let $C[0,2]$ be the linear space of continuous functions $f : [0,2] \to \mathbb{R}$. Consider $C[0,2]$ equipped with the norm $\|f\|_{L^1} = \int_{[0,2]} |f|$. For $n \in \mathbb{N}$, let $f_n : [0,2] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{4}), \\ n(x - \frac{1}{4}), & x \in [\frac{1}{4}, \frac{1}{4} + \frac{1}{n}), \\ 1, & x \in [\frac{1}{4} + \frac{1}{n}, 2]. \end{cases}$$

(i) Prove that $f_n \in C[0,2]$ and that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(C[0,2], \|\cdot\|_{L^1})$. **[5]**

(ii) Is $(C[0,2], \|\cdot\|_{L^1})$ a Banach space? Give a full justification of your response. **[8]**

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8. (a) Let $M \subset \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$ with $M \neq \mathcal{H}$. Denote by $P : \mathcal{H} \to M$ the orthogonal projection of $\mathcal{H}$ onto M, i.e., if $x \in \mathcal{H}$, then $Px \in M$ is the unique closest point to x in M.

Let $x \in \mathcal{H}$. Suppose that $y \in M$ is such that $(x - y) \perp M$. Prove that $y = Px$. **[5]**

(b) Recall that $\{e^{ikx} : k \in \mathbb{Z}\}$ is an orthonormal set in the Hilbert space $L^2[-\pi, \pi]$ equipped with the inner product for $f, g \in L^2[-\pi, \pi]$:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi,\pi]} f\overline{g}, \quad f, g \in L^2[-\pi, \pi].$$

Recall that the Fourier coefficients of $f \in L^2[-\pi, \pi]$ are defined as

$$a_k(f) = \langle f, e^{iky} \rangle = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(y)e^{-iky}.$$

Let $n \in \mathbb{N}$. By using the fact that

$$0 \leq \frac{1}{2\pi} \int_{[-\pi,\pi]} \left| f - \sum_{k=-n}^{n} a_k(f)e^{ikx} \right|^2,$$

prove that

$$\sum_{k \in \mathbb{Z}} |a_k(f)|^2 \leq \int |f|^2.$$ **[10]**

END