Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3021-WE01

Title:

Differential Geometry III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{R}^3$ be a smooth curve parametrised by arc length, which satisfies $\boldsymbol{\alpha}''(s) \neq 0$.

(a) Give the definition of the unit tangent vector $\mathbf{t}(s)$ and the curvature $\kappa(s)$. **[2]**

(b) Give the definition of the principal normal vector $\mathbf{n}(s)$ and the binormal vector $\mathbf{b}(s)$. **[2]**

(c) Show that $\mathbf{b}'(s)$ is orthogonal to both $\mathbf{t}(s)$ and $\mathbf{b}(s)$, and define the torsion $\tau(s)$. **[3]**

(d) Show that $\mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s)$. **[3]**

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2. Let U be an open set in $\mathbb{R}^3$ and let $f : U \longrightarrow \mathbb{R}$ be a smooth function.

(a) For $c \in \mathbb{R}$ give a criterion involving the gradient $\nabla(f)$ that ensures $f^{-1}(c) = \big\{\mathbf{p} \in U : f(\mathbf{p}) = c\big\}$ is a regular surface. **[2]**

(b) Use the criterion from (a) to show that the graph of a smooth function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a regular surface. **[3]**

(c) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by $f(x, y, z) = x^3y^3 - z^3$. Find all $c \in \mathbb{R}$ for which $f^{-1}(c)$ is a regular surface. **[5]**

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3. Let $U = \big\{(u, v) \in \mathbb{R}^2 : v > 0\big\}$ be the *hyperbolic plane* with the first fundamental form given by

$$E(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0, \quad G(u, v) = \frac{1}{v^2}.$$

(a) Let $\boldsymbol{\alpha} : \mathbb{R} \longrightarrow U$ be given by $\boldsymbol{\alpha}(t) = \big(u(t), v(t)\big)$. Show that $\boldsymbol{\alpha}$ is a geodesic if

$$u''(t) = \frac{2u'(t)v'(t)}{v(t)}, \quad v''(t) = \frac{v'(t)^2 - u'(t)^2}{v(t)}.$$ **[5]**

(b) Show that the curves $\boldsymbol{\alpha} : \mathbb{R} \longrightarrow U$ and $\boldsymbol{\beta} : \mathbb{R} \longrightarrow U$ are geodesics, where

$$\boldsymbol{\alpha}(t) = (0, e^t), \quad \boldsymbol{\beta}(t) = \left(\frac{\sinh(t)}{\cosh(t)}, \frac{1}{\cosh(t)}\right).$$ **[5]**

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4. Let $\mathbf{x} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $\mathbf{x}(u, v) = (u, v, uv)$ be a parametrisation of the *hyperbolic paraboloid.*

(a) Find the coefficients of the second fundamental form of $\mathbf{x}$. **[5]**

(b) Find the asymptotic curves on the hyperbolic paraboloid. **[5]**

SECTION B

5. The *deltoid* is a plane curve with parametrisation $\boldsymbol{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}^2$ given by

$$\boldsymbol{\alpha}(u) = \big(\cos(2u) + 2\cos(u),\ -\sin(2u) + 2\sin(u)\big).$$

(a) Show that the deltoid has vertices at $\boldsymbol{\alpha}(2k\pi/3)$ for all $k \in \mathbb{Z}$. **[3]**

(b) Find the arc length of the deltoid between the vertices $\boldsymbol{\alpha}(0)$ and $\boldsymbol{\alpha}(2\pi/3)$. **[2]**

(c) Show that the curvature at $\boldsymbol{\alpha}(u)$ is

$$\kappa(u) = \frac{-1}{8|\sin(3u/2)|}.$$ **[5]**

(d) Show that the evolute of the deltoid has parametrisation

$$\mathbf{e}(u) = \big(-3\cos(2u) + 6\cos(u),\ 3\sin(2u) + 6\sin(u)\big).$$ **[5]**

Hint: you may find it useful to use

$$\cos(2u) - \cos(u) = -2\sin(3u/2)\sin(u/2),\ \sin(2u) + \sin(u) = 2\sin(3u/2)\cos(u/2).$$

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6. Let $\boldsymbol{\alpha}(u)$ be a curve parametrised by arc length u. For $r > 0$ sufficiently small consider the *canal surface* S with parametrisation

$$\mathbf{x}(u, v) = \boldsymbol{\alpha}(u) + r\cos(v)\mathbf{n}(u) + r\sin(v)\mathbf{b}(u).$$

(a) Express the partial derivatives $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ in terms of the moving frame basis $\big\{\mathbf{t}(u), \mathbf{n}(u), \mathbf{b}(u)\big\}$. **[4]**

(b) Show that the partial derivatives of $\mathbf{x}$ satisfy

$$\mathbf{x}_u(u, v) + \tau(u)\mathbf{x}_v(u, v) = \big(1 - r\kappa(u)\cos(v)\big)\mathbf{t}(u).$$ **[3]**

(c) Deduce that S is not a regular surface if $r = 1/\kappa(u)$. **[2]**

(d) Calculate the coefficients of the first fundamental form of $\mathbf{x}$. **[6]**

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7. Let $U = \{(u,v) \in \mathbb{R}^2 : 0 < u < 2\pi,\ v > 0\}$. A local parametrisation $\mathbf{x} : U \longrightarrow \mathbb{R}^3$ of the *pseudosphere* is given by

$$\mathbf{x}(u,v) = \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, v - \frac{\sinh(v)}{\cosh(v)}\right).$$

(a) Find the coefficients of the first fundamental form for $\mathbf{x}$. **[4]**

(b) Show that the pseudosphere has area 2π. **[3]**

(c) Find the coefficients of the second fundamental form for $\mathbf{x}$. **[4]**

(d) Show that the pseudosphere has constant Gaussian curvature $K = -1$. **[4]**

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8. The *hyperboloid of one sheet* S is given as a level set by

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}.$$

(a) Show that S admits the following local parametrisation as a ruled surface

$$\mathbf{x}(u,v) = \big(\cos(u), \sin(u), 0\big) + v\big(-\sin(u), \cos(u), 1\big).$$ **[2]**

(b) Show that a unit normal to S at $\mathbf{p} = \mathbf{x}(u,v)$ is given by

$$\mathbf{N}_{\mathbf{p}} = \frac{1}{\sqrt{1+2v^2}}\big(\cos(u) - v\sin(u), \sin(u) + v\cos(u), -v\big).$$ **[4]**

(c) For fixed u_0 show that the curve $\boldsymbol{\alpha}_{u_0}(v) = \mathbf{x}(u_0, v)$ is a geodesic. **[3]**

(d) For fixed v_0 find the geodesic curvature of the curve $\boldsymbol{\beta}_{v_0}(u) = \mathbf{x}(u, v_0)$. Conclude that $\boldsymbol{\beta}_{v_0}$ is a geodesic if and only if $v_0 = 0$. **[6]**
