Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3031-WE01

Title: Number Theory III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) Show that there is no solution in integers other than $x = y = z = 0$ to

$$3x^{11} + 5y^{11} = 7z^{11}\,.$$

[**5**]

(b) For $m \in \mathbb{Z}$ show that

$$\frac{m - \sqrt[3]{m}}{\sqrt[3]{3}}$$

is an algebraic integer. [**5**]

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2. Let $K = \mathbb{Q}(\sqrt{-23})$ and let $R = \mathcal{O}_{-23}$ be its ring of integers.

(a) Show that the ideal $J = (1 + 5\sqrt{-23}, 1 + \sqrt{-23})_R \subset R$ is not a principal ideal. [**4**]

(b) Compute the ideal J^2 and write it with at most two generators, where one of the generators is an integer. [**2**]

(c) Put $I = (\frac{1+\sqrt{-23}}{2}, 3)_R$. Show that $R/I \simeq \mathbb{Z}/3$. (Carefully state any theorem that you use.) [**4**]

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3. Let $\theta \in \mathbb{C}$ be a root of $f(x) = x^3 + 2x + 2$ and let $K = \mathbb{Q}(\theta)$. In this question, you may assume that $R = \mathbb{Z}[\theta]$ is the ring of integers of K.

(a) Find $[K : \mathbb{Q}]$ and compute $N_K(4 - \theta)$. [**3**]

(b) Find two distinct prime ideals of R (given in terms of generators) that divide $(N_K(4 - \theta))_R$. [**4**]

(c) Factorise the ideal $(4 - \theta)_R$ as a product of prime ideals. [**3**]

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4. (a) Let K be a number field and let I be a non-zero ideal of $\mathcal{O}_K$. Prove that there are only finitely many ideals of $\mathcal{O}_K$ that contain I. **[5]**

(b) Assume that R is an integral domain such that for any ideal I of R, there exists an $n \in \mathbb{Z}$, $n \geq 1$, such that I^n is principal. Show that for ideals of R, "to contain is to divide", that is, for any two ideals I and J of R, if $I \subseteq J$, then there is an ideal S of R such that $I = JS$.
(Hint: Consider an ideal S of the form $(\frac{1}{\alpha})IJ^{n-1}$, for some $n \in \mathbb{Z}$, $n \geq 1$ and $\alpha \in R$. You must show that S is an ideal.) **[5]**

SECTION B

5. (a) Give the ring of integers $\mathcal{O}_{61}$ in $\mathbb{Q}(\sqrt{61})$ and find its fundamental unit. **[5]**

(b) Find positive integers a, b such that

$$a^2 - 61b^2 = 4.$$

(Note the positive sign on the right.) **[3]**

(d) Give all the solutions in integers x, y, if any, to the equation

$$x^2 - 18y^2 = 7.$$

[7]

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6. Let $R = \mathbb{Z}[\sqrt{-2}]$.

(a) Show that R is a Euclidean domain. **[5]**

(b) Write 11 and 59 in $\mathbb{Z}[\sqrt{-2}]$ as a product of irreducible elements in all possible ways, up to associates. **[2]**

(c) Determine the number of different $\alpha \in R$ dividing 649. **[3]**

(d) Using the previous parts, or otherwise, find all the solutions to the equation **[5]**

$$a^2 + 2b^2 = 649$$

in integers.

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7. Let $\theta \in \mathbb{C}$ be such that $\theta^3 = 9$ and let $K = \mathbb{Q}(\theta)$.

(a) Find the discriminant $\Delta_K(1, \theta, \theta^2)$. **[3]**

(b) Show that $\mathbb{Z}[\theta^2/3]$ is a full lattice in K by finding a generating basis. Find the discriminant $\Delta_K(\mathbb{Z}[\theta^2/3])$. **[4]**

(c) Show that $\theta^2/3 \in \mathcal{O}_K$ and that $\mathcal{O}_K = \mathbb{Z}[\theta^2/3]$. **[8]**

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8. Let $\theta = \sqrt{-30}$, $K = \mathbb{Q}(\theta)$ and $R = \mathcal{O}_K$.

(a) Determine the class group $Cl(R)$. You may use the Minkowski bound, given by $B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n}\sqrt{|\Delta_K|}$. **[6]**

(b) Let $\mathfrak{p}_{11}$ be any prime ideal of R above 11. In your computation of $Cl(R)$ in the previous part, find the ideal class in $Cl(R)$ that $\mathfrak{p}_{11}$ belongs to. In other words, find the element of $Cl(R)$ that is equal to $[\mathfrak{p}_{11}]$. **[9]**
