



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH30320-WE01
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Title: Differential Geometry V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a smooth curve parametrised by arc length, which satisfies $\alpha''(s) \neq 0$.
- (a) Give the definition of the unit tangent vector $\mathbf{t}(s)$ and the curvature $\kappa(s)$. [2]
- (b) Give the definition of the principal normal vector $\mathbf{n}(s)$ and the binormal vector $\mathbf{b}(s)$. [2]
- (c) Show that $\mathbf{b}'(s)$ is orthogonal to both $\mathbf{t}(s)$ and $\mathbf{b}(s)$, and define the torsion $\tau(s)$. [3]
- (d) Show that $\mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s)$. [3]
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2. Let U be an open set in \mathbb{R}^3 and let $f : U \rightarrow \mathbb{R}$ be a smooth function.
- (a) For $c \in \mathbb{R}$ give a criterion involving the gradient $\nabla(f)$ that ensures $f^{-1}(c) = \{\mathbf{p} \in U : f(\mathbf{p}) = c\}$ is a regular surface. [2]
- (b) Use the criterion from (a) to show that the graph of a smooth function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a regular surface. [3]
- (c) Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by $f(x, y, z) = x^3y^3 - z^3$. Find all $c \in \mathbb{R}$ for which $f^{-1}(c)$ is a regular surface. [5]
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3. Let $U = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ be the *hyperbolic plane* with the first fundamental form given by

$$E(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0, \quad G(u, v) = \frac{1}{v^2}.$$

- (a) Let $\alpha : \mathbb{R} \rightarrow U$ be given by $\alpha(t) = (u(t), v(t))$. Show that α is a geodesic if

$$u''(t) = \frac{2u'(t)v'(t)}{v(t)}, \quad v''(t) = \frac{v'(t)^2 - u'(t)^2}{v(t)}. \quad [5]$$

- (b) Show that the curves $\alpha : \mathbb{R} \rightarrow U$ and $\beta : \mathbb{R} \rightarrow U$ are geodesics, where

$$\alpha(t) = (0, e^t), \quad \beta(t) = \left(\frac{\sinh(t)}{\cosh(t)}, \frac{1}{\cosh(t)} \right). \quad [5]$$

4. Let $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $\mathbf{x}(u, v) = (u, v, uv)$ be a parametrisation of the *hyperbolic paraboloid*.

- (a) Find the coefficients of the second fundamental form of \mathbf{x} . [5]
- (b) Find the asymptotic curves on the hyperbolic paraboloid. [5]
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SECTION B

5. The *deltoid* is a plane curve with parametrisation $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ given by

$$\alpha(u) = (\cos(2u) + 2\cos(u), -\sin(2u) + 2\sin(u)).$$

- (a) Show that the deltoid has vertices at $\alpha(2k\pi/3)$ for all $k \in \mathbb{Z}$. [3]
 (b) Find the arc length of the deltoid between the vertices $\alpha(0)$ and $\alpha(2\pi/3)$. [2]
 (c) Show that the curvature at $\alpha(u)$ is

$$\kappa(u) = \frac{-1}{8|\sin(3u/2)|}. \quad [5]$$

- (d) Show that the evolute of the deltoid has parametrisation

$$\mathbf{e}(u) = (-3\cos(2u) + 6\cos(u), 3\sin(2u) + 6\sin(u)). \quad [5]$$

Hint: you may find it useful to use

$$\cos(2u) - \cos(u) = -2\sin(3u/2)\sin(u/2), \quad \sin(2u) + \sin(u) = 2\sin(3u/2)\cos(u/2).$$

6. Let $\alpha(u)$ be a curve parametrised by arc length u . For $r > 0$ sufficiently small consider the *canal surface* S with parametrisation

$$\mathbf{x}(u, v) = \alpha(u) + r\cos(v)\mathbf{n}(u) + r\sin(v)\mathbf{b}(u).$$

- (a) Express the partial derivatives $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ in terms of the moving frame basis $\{\mathbf{t}(u), \mathbf{n}(u), \mathbf{b}(u)\}$. [4]
 (b) Show that the partial derivatives of \mathbf{x} satisfy

$$\mathbf{x}_u(u, v) + \tau(u)\mathbf{x}_v(u, v) = (1 - r\kappa(u)\cos(v))\mathbf{t}(u). \quad [3]$$

- (c) Deduce that S is not a regular surface if $r = 1/\kappa(u)$. [2]
 (d) Calculate the coefficients of the first fundamental form of \mathbf{x} . [6]
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7. Let $U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi, v > 0\}$. A local parametrisation $\mathbf{x} : U \rightarrow \mathbb{R}^3$ of the *pseudosphere* is given by

$$\mathbf{x}(u, v) = \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, v - \frac{\sinh(v)}{\cosh(v)} \right).$$

- (a) Find the coefficients of the first fundamental form for \mathbf{x} . [4]
 (b) Show that the pseudosphere has area 2π . [3]
 (c) Find the coefficients of the second fundamental form for \mathbf{x} . [4]
 (d) Show that the pseudosphere has constant Gaussian curvature $K = -1$. [4]
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8. The *hyperboloid of one sheet* S is given as a level set by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}.$$

- (a) Show that S admits the following local parametrisation as a ruled surface

$$\mathbf{x}(u, v) = (\cos(u), \sin(u), 0) + v(-\sin(u), \cos(u), 1). \quad [2]$$

- (b) Show that a unit normal to S at $\mathbf{p} = \mathbf{x}(u, v)$ is given by

$$\mathbf{N}_{\mathbf{p}} = \frac{1}{\sqrt{1+2v^2}}(\cos(u) - v\sin(u), \sin(u) + v\cos(u), -v). \quad [4]$$

- (c) For fixed u_0 show that the curve $\boldsymbol{\alpha}_{u_0}(v) = \mathbf{x}(u_0, v)$ is a geodesic. [3]
 (d) For fixed v_0 find the geodesic curvature of the curve $\boldsymbol{\beta}_{v_0}(u) = \mathbf{x}(u, v_0)$.
 Conclude that $\boldsymbol{\beta}_{v_0}$ is a geodesic if and only if $v_0 = 0$. [6]
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