Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH30420-WE01

Title:
Galois Theory V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $L = \mathbb{F}_{729}$ be the field with 729 elements.

(a) List the prime subfield and all proper subfields of L. **[3]**

(b) For one of these proper subfields, say K, find a polynomial over the prime subfield whose roots generate K. **[4]**

(c) Determine all the odd primes p for which the field L contains a primitive p-th root of unity. **[3]**

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2. Consider the polynomial $f(x) = x^3 - x^2 + 1 \in \mathbb{F}_3[x]$ where $\mathbb{F}_3$ is the field with 3 elements.

(a) Show that $f(x)$ is irreducible over $\mathbb{F}_3$. **[2]**

(b) Given a root θ of $f(x)$ in some field extension of $\mathbb{F}_3$, express the other two roots of $f(x)$ in the form $a + b\theta + c\theta^2$ where $a, b, c \in \mathbb{F}_3$. **[4]**

(c) Does the element θ generate the multiplicative subgroup of the field $\mathbb{F}_3(\theta)$? Justify your answer. **[4]**

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3. Consider the field $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{2} + i$.

(a) Describe the structure of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. **[4]**

(b) Find the minimal polynomial of α over $\mathbb{Q}$. **[2]**

(c) Show that $\theta = \sqrt[3]{\alpha}$ is not an element of K and hence determine the minimal polynomial of θ over $\mathbb{Q}$. **[4]**

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4. Find the Galois group of the normal closure of each of the following polynomials in $\mathbb{Q}[x]$.

(a) $x^3 - 21x - 35$ **[3]**

(b) $x^4 + 3x^2 - 6x + 10$ **[2]**

(c) $x^4 - 7x^2 - 3x + 1$ **[5]**

SECTION B

5. Suppose L is a splitting field of the polynomial $x^6 - 7x^3 + 10$ over $\mathbb{Q}$ and let $K = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/3}$.

(a) Given a rational number r with real cube root $\sqrt[3]{r} \notin \mathbb{Q}$, explain why we also have $\sqrt[3]{r} \notin K$. **[4]**

(b) Show that $K \subset L$ and determine the degree $[L : K]$. **[6]**

(c) Describe the structure of the Galois group $\mathrm{Gal}(L/K)$, and find all intermediate fields M with $K \subset M \subset L$. **[5]**

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6. For any $n \in \mathbb{N}$, let ζ_n be an n-th primitive root of unity.

(a) Prove that $\{\zeta_7^m \mid m = 1, 2, 3, 4, 5, 6\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta_7)$ (note the unusual choice). **[2]**

(b) Find rational numbers a_j for $1 \le j \le 6$, such that

$$\sqrt{-7} = \sum_{j=1}^{6} a_j \zeta_7^j \, .$$ **[5]**

(c) Let $L = \mathbb{Q}(\zeta_{84})$. Prove that $\sqrt{21} \in L$, and find the structure of the group $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{21}))$. **[4]**

(d) How many subfields N such that $[N : \mathbb{Q}] = 2$ are there in L? List all these fields N in the form $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$. **[4]**

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7. (a) Write down both the quadratic resolvent of a reduced cubic polynomial and the cubic resolvent of a reduced quartic polynomial. **[3]**

(b) By finding the cubic resolvent, or otherwise, find the roots of the quartic polynomial

$$x^4 - 7x^2 + 8x - \frac{7}{4}.$$ **[6]**

(c) For an odd integer n, consider

$$f_n(x) = x^4 - (n^2 + 4)x^2 + (n^2 + 4) \in \mathbb{Q}[x].$$

Show that the splitting field of $f_n(x)$ is a cyclic Galois extension of $\mathbb{Q}$. **[6]**

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8. Let L be a splitting field of $f(x) = x^4 - 4x^2 - 3$ over $\mathbb{Q}$ and let $K = \mathbb{Q}(\sqrt{-3})$.

(a) Prove that $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the dihedral group D_4 with 8 elements. **[5]**

(b) Show that $K \subset L$ and construct generators of $\mathrm{Gal}(L/K)$, giving their action on the roots of $f(x)$. **[5]**

(c) Find elements β and γ in K such that $L = K(\sqrt{\beta}, \sqrt{\gamma})$. **[5]**
