Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3071-WE01

Title:
Decision Theory III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Suppose we wish to estimate an unknown quantity x, which is modelled as a continuous random variable with density f_X defined on $\mathbb{R}$. Assume that X has a finite mean and variance, and let $z \in \mathbb{R}$ denote a decision (that is, an estimate of x).

(a) Give the definitions of the Bayes decision and the Bayes risk corresponding to an arbitrary loss function $L : \mathbb{R}^2 \to \mathbb{R}$. **[3]**

(b) Suppose the loss is squared error:

$$L(x, z) = (x - z)^2.$$

Find the Bayes decision and Bayes risk. **[2]**

(c) Now suppose that the loss has the form

$$L(x, z) = g(x)(x - z)^2,$$

for some function g which satisfies $g(x) > g(-x) > 0$ for all $x > 0$. Assume also that $f_X(x) = f_X(-x)$ for all $x \geq 0$. Show that the Bayes decision, if it exists, is strictly positive. **[5]**

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2. Suppose that a vegetable patch may be planted with proportions $(p_C, p_T, p_R, p_W, p_B)$ of carrots (C), tomatoes (T), radishes (R), watercress (W), and basil (B), where $p_C + p_T + p_R + p_W + p_B = 1$. We interpret these proportions as probabilities in a lottery over the five vegetable types. Our preferences over possible gardens satisfy the assumptions of the Von Neumann-Morgenstern utility theorem.

We have the following information about our preferences:

- Amongst gardens with only one vegetable, we prefer radishes most, and carrots least.
- We are indifferent between a garden containing 60% carrots and 40% radishes and a garden containing only tomatoes.
- We are indifferent between a garden containing 50% watercress, 20% basil and 30% carrots and a garden containing 30% watercress, 30% basil, 20% tomatoes and 20% carrots.
- We are indifferent between a garden containing 60% watercress, 20% basil and 20% carrots and a garden containing 40% watercress, 40% basil, 15% tomatoes and 5% carrots.

(a) Find a formula for utility of a garden with proportions $(p_C, p_T, p_R, p_W, p_B)$ of C, T, R, W, and B respectively, consistent with the above preferences and assumptions. Scale the utility to have maximum value 1 and minimum value 0. **[6]**

(b) Suppose that the total area of our garden is 5 m^2. The cost to plant each vegetable is

Vegetable	C	T	R	W	B
Cost per m^2 (£)	0	2	5	3	4

Any unplanted area is filled with carrots, at no cost. We have a total budget of £10. How should we plant the patch to maximise utility subject to our budget and area limits? **[4]**

3. (a) Consider the decision problem with pay-off table given below. You have the choice between four actions $(a_1, \ldots, a_4)$ and there are four unknown states $(\theta_1, \ldots, \theta_4)$.

	θ_1	θ_2	θ_3	θ_4
a_1	1	2	3	1
a_2	3	1	1	2
a_3	4	0	0	1
a_4	1	3	1	0

Assume that your uncertainty about the states is reflected by the following probabilities: $P(\theta_1) = P(\theta_2) = 0.4$ and $P(\theta_3) = P(\theta_4) = 0.1$.

Determine the optimal actions according to the following criteria: (1) maximisation of expected pay-off; (2) maximisation of minimum pay-off; (3) minimisation of maximum regret. **[5]**

(b) Consider a two-person game where each player has four possible strategies: $R1, R2, R3, R4$ for Player 1; $C1, C2, C3, C4$ for Player 2. The following pay-offs correspond to each pair of strategies, where e.g. (2,6) denotes pay-off 2 to Player 1 and pay-off 6 to Player 2.

	$C1$	$C2$	$C3$	$C4$
$R1$	(2,1)	(2,6)	(4,0)	(0,3)
$R2$	(2,5)	(3,3)	(1,5)	(3,0)
$R3$	(1,0)	(2,2)	(0,1)	(4,7)
$R4$	(0,4)	(1,3)	(4,2)	(1,1)

Find all pure Nash equilibria for this game. Discuss briefly whether or not it is in a player's interest to play a strategy corresponding to a pure Nash equilibrium; illustrate the discussion using this example. **[5]**

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4. Consider the zero-sum game in which R chooses strategy (row) $R1$ or $R2$, C chooses strategy (column) $C1, C2, C3, C4$ or $C5$, and the payoffs to R are as follows

	$C1$	$C2$	$C3$	$C4$	$C5$
$R1$	-2	5	6	3	0
$R2$	5	2	0	1	3

The payoff to C is minus the payoff to R.

(a) Use a graphical method to identify the minimax strategies for R and for C, and the value of the game. **[6]**

(b) Suppose that each player in this game decides on their strategy by minimising their maximum regret. Determine their strategies and the resulting payoff. **[4]**

SECTION B

5. We are running a (small) company that makes toothbrushes and toothpaste. We are planning on expanding operations over the coming year. At time 0 (now), we can either choose to expand our toothbrush production (B) costing £5 or our toothpaste production (P), costing £10. The expansion takes one year.

During that year, we observe 10 independent sales, each of which can be either a toothbrush, with probability ρ, or toothpaste, where $\rho \in [0, 1]$ is unknown. If we choose B, once we have finished the year, we may market our product (M) or not (N). Marketing costs £10. If we choose P, we cannot then choose to market.

If we choose B and choose to market, our income (in £) is $100\rho^2$. If we choose B and do not market, our income is $50\rho^2$. If we choose P, our income is 60ρ. Our profit is our income less any costs of expansion or marketing.

(a) Suppose we model our prior belief about ρ by a Beta distribution $\text{Beta}(\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$. Show that after observing $T = t$ toothbrush purchases (out of 10 sales) the posterior distribution of ρ is $\text{Beta}(t + \alpha, 10 - t + \beta)$. You must prove any necessary results regarding conjugate priors. **[4]**

(b) Suppose that we consider that ρ is equally likely to take any value in $[0, 1]$. Find the decision rule (the product in which to expand, and after observing t, whether to market) which maximises our expected profit. You may assume without proof that $P(T = t) = 1/11$ for any $t \in \{0, 1, \dots 10\}$. The following formula may also be of use:

$$\sum_{i=0}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2).$$ **[11]**

Note: if $X \sim \text{Beta}(\alpha, \beta)$ (the Beta distribution) we have

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{C(\alpha, \beta)}, \text{ and } E(X) = \frac{\alpha}{\alpha+\beta}, \ \text{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

where $C(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. If $Y \sim \text{Bin}(n, y)$ (the binomial distribution) we have:

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}.$$

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6. Suppose that for two attributes (x, y) taking values in $\mathbb{R}^2$, we have a joint utility function given by:

$$U(x, y) = x^2(1 + y)^2$$

up to a multiplicative and additive constant.

(a) Suppose we consider origin $(x_0, y_0) = (1, 0)$. Show that our utility function is consistent with the attributes being mutually utility independent. **[4]**

(b) Suppose we instead use an arbitrary origin (x_0, y_0). Find an expression for the corresponding value of K in terms of (x_0, y_0), if such a value exists. **[6]**

(c) Suppose we are instead given an arbitrary joint utility function $U(x, y)$ over mutually utility independent attributes x, y taking values in the real numbers. Suppose that U is twice differentiable, and that for all $(x, y) \in \mathbb{R}^2$ we have:

$$\frac{\partial}{\partial x}U(x, y) > 0 \text{ and } \frac{\partial}{\partial y}U(x, y) > 0.$$

Show that the attributes are complementary if and only if:

$$\frac{\partial^2}{\partial x \partial y}U(x, y) \geq 0,$$

for all $(x, y) \in \mathbb{R}^2$. **[5]**

7. (a) The individual utilities of 4 people for 5 options are given in the table below.

	option				
person	a	b	c	d	e
A	0	0	10	3	5
B	0.3	0	0	1	0.6
C	2	-1	3	-1	0
D	20	-5	-5	10	15

(i) Find the group's preference ordering over these options according to Harsanyi's Utilitarianism theorem.

(ii) Illustrate briefly how person A could manipulate the resulting group preference ordering if he knew the utilities of the other 3 people in advance. Also comment on the possibility for A to manipulate the result if the aim for the group is to choose a single option as winner, with the ordering over the other options irrelevant.

(iii) Suppose that option b is no longer available. Discuss whether or not the method in Harsanyi's Utilitarianism theorem can still be applied to combine the individual preferences into a group preference. **[9]**

(b) Consider the following procedure to combine preference orderings of $m \geq 3$ people over $k \geq 3$ options, where everybody can express their preferences without restrictions:

The group decides to adopt the preference ordering of the oldest person in the group, except if at least 60% of the group members disagree with any of the pairwise preferences of the oldest person. If there is such a disagreement, then the group adopts the preference ordering of the youngest person in the group. Assume that the group does not contain people who are born on the same day.

Consider this procedure for combining a group preference profile into a single group preference ordering from the perspective of Arrow's theory of Social Choice. Explain in detail for each axiom whether or not it is satisfied by this procedure. Include an example to illustrate each axiom which is not satisfied. **[6]**

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8. Consider a bargaining problem with 7 options, for which the utilities (u to first person, v to second person) are given in the following table:

	1	2	3	4	5	6	7
u	-1	-1	2	3	5	6	6
v	-1	7	7	5	4	3	1

The utility pair $(-1, -1)$ is the status-quo point for this problem.

(a) Sketch the feasible region and give the equation(s) for the Pareto boundary for this bargaining problem. **[2]**

(b) Derive the Nash point. Specify all bargains the players can agree on corresponding to the Nash point. **[6]**

(c) Derive the equitable distribution point and specify a corresponding bargain. **[4]**

(d) Suppose that the status-quo point changes to $(2, 2)$. Without further computations, explain how this affects the Nash point. Mention any axioms you use in your explanation. **[3]**