Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH30720-WE01

Title:
Dynamical Systems V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

1. A planar system is given by

$$\dot{x} = (1 - r^2)(2 - r)\,x - r\,y,$$
$$\dot{y} = (1 - r^2)(2 - r)\,y + r\,x,$$
$$r = \sqrt{x^2 + y^2}.$$

(a) Rewrite the system in polar coordinates (r, θ) with $x = r\cos\theta$ and $y = r\sin\theta$. **[3]**

(b) Show that r satisfies a one-dimensional autonomous ODE $\dot{r} = f(r)$ and find $f(r)$. **[2]**

(c) Find all equilibria of the reduced 1D system for $r \geq 0$ and classify their stability. **[2]**

(d) Sketch the phase flow in the plane and describe the long-time behaviour for different initial radii. **[3]**

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2. A two-dimensional linear system is given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x} = (x, y)^T,$$

where

$$A = \begin{pmatrix} -8 & 7 \\ -9 & 8 \end{pmatrix}.$$

(a) Explicitly find a similarity matrix M and use it to put A into Jordan normal form $J = M^{-1}AM$ and use this to write the explicit solution $\mathbf{x}(t)$ in the original coordinates. **[7]**

(b) Sketch the phase flow and identify the stable and unstable invariant lines through the origin. **[3]**

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3. Sketch the bifurcation diagrams, including some representative trajectories, for each of the following systems:

(a) $x' = x^3 - \mu x$. **[5]**

(b) $x' = \mu - \sin x$. (Consider $x \in [-\pi, \pi]$.) **[5]**

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4. (a) Define what is meant by the ω-*limit set* of a point p in phase space. **[2]**

(b) Sketch the phase portrait of the system

$$x' = -y \quad \text{and} \quad y' = x - x^3$$

clearly labelling all fixed points. **[4]**

(c) Write down all ω-limit sets in this system, clearly noting which point(s) they apply to. **[4]**

SECTION B

5. A two-dimensional system is defined as

$$\dot{x} = x + y^2 - 1,$$
$$\dot{y} = 1 - x^2 - y.$$

(a) State what it means that a dynamical system is Hamiltonian and show that this system is Hamiltonian. **[2]**

(b) Find all equilibrium points for the system. **[3]**

(c) By linearisation, determine the nature of each equilibrium point. **[4]**

(d) Prove that H is conserved along trajectories. Show that the two saddle points have the same energy E_* and write the level set $H(x, y) = E_*$ explicitly. Explain why this level set contains the separatrix through the saddles. **[4]**

(e) Sketch the phase portrait, indicating the saddles, centers, and the separatrix curve. **[2]**

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6. A two-dimensional dynamical system is given by

$$\dot{x} = y,$$
$$\dot{y} = x(1 - x^4).$$

(a) Solve the system implicitly to find the equations of the phase paths $y = y(x)$. **[3]**

(b) Find all equilibria and classify them using linearisation. **[3]**

(c) State the Stable Manifold Theorem for a generic planar system near a hyperbolic fixed point. **[2]**

(d) Apply the Stable Manifold Theorem at $(0, 0)$ and *verify it explicitly* by finding tangent vectors from the nonlinear phase-curve equations and showing that the dimension statement holds. **[3]**

(e) Sketch the phase portrait and describe the qualitative behaviour near each equilibrium. **[4]**

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7. This question concerns the damped pendulum,

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -\sin x - \gamma y,$$

for some constant $\gamma \geq 0$.

(a) For $\gamma = 0$, find a first integral of the system. **[3]**

(b) Prove that the system cannot have closed orbits if $\gamma > 0$, citing any results you use (no need to prove them). **[4]**

(c) Define what is meant by a *Lyapunov function* of a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$. **[3]**

(d) For small $\gamma > 0$, find a Lyapunov function for this system and use it to prove that, except for two orbits ending at the top point $(\pm\pi, 0)$, all solutions tend to the origin. **[5]**

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8. Consider the following system

$$\dot{x} = -y + x^3 - x^5 \quad \text{and} \quad \dot{y} = x + y^3 - y^5.$$

The following identities may be useful: $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta$, $\sin^4\theta + \cos^4\theta = \frac{1}{4}(3 + \cos 4\theta)$ and $\sin^6\theta + \cos^6\theta = \frac{1}{8}(5 + 3\cos 4\theta)$.

(a) Rewrite the system in polar coordinates. **[3]**

(b) Define what is meant by a *positively invariant set*. **[2]**

(c) For the system above, find $R_1 > 0$ and $R_2 < \infty$ such that the annulus $\{R_1 \leq r \leq R_2\}$ is positively invariant. **[4]**

(d) State (do not prove) Poincaré–Bendixson Theorem. **[3]**

(e) Assuming that the above system has a single fixed point, what can you say about its possible ω-limit sets? **[3]**
