



## EXAMINATION PAPER

<b>Examination Session:</b> May/June	<b>Year:</b> 2026	<b>Exam Code:</b> MATH3091-WE01
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<b>Title:</b> Dynamical Systems III
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Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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<b>Revision:</b>	
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## SECTION A

1. A planar system is given by

$$\begin{aligned}\dot{x} &= (1 - r^2)(2 - r)x - ry, \\ \dot{y} &= (1 - r^2)(2 - r)y + rx, \\ r &= \sqrt{x^2 + y^2}.\end{aligned}$$

- (a) Rewrite the system in polar coordinates  $(r, \theta)$  with  $x = r \cos \theta$  and  $y = r \sin \theta$ . [3]
- (b) Show that  $r$  satisfies a one-dimensional autonomous ODE  $\dot{r} = f(r)$  and find  $f(r)$ . [2]
- (c) Find all equilibria of the reduced 1D system for  $r \geq 0$  and classify their stability. [2]
- (d) Sketch the phase flow in the plane and describe the long-time behaviour for different initial radii. [3]
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2. A two-dimensional linear system is given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} = (x, y)^T,$$

where

$$A = \begin{pmatrix} -8 & 7 \\ -9 & 8 \end{pmatrix}.$$

- (a) Explicitly find a similarity matrix  $M$  and use it to put  $A$  into Jordan normal form  $J = M^{-1}AM$  and use this to write the explicit solution  $\mathbf{x}(t)$  in the original coordinates. [7]
- (b) Sketch the phase flow and identify the stable and unstable invariant lines through the origin. [3]
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3. Sketch the bifurcation diagrams, including some representative trajectories, for each of the following systems:

- (a)  $x' = x^3 - \mu x$ . [5]
- (b)  $x' = \mu - \sin x$ . (Consider  $x \in [-\pi, \pi]$ .) [5]
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4. (a) Define what is meant by the  $\omega$ -limit set of a point  $p$  in phase space. [2]  
(b) Sketch the phase portrait of the system

$$x' = -y \quad \text{and} \quad y' = x - x^3$$

clearly labelling all fixed points. [4]

- (c) Write down all  $\omega$ -limit sets in this system, clearly noting which point(s) they apply to. [4]
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## SECTION B

5. A two-dimensional system is defined as

$$\begin{aligned}\dot{x} &= x + y^2 - 1, \\ \dot{y} &= 1 - x^2 - y.\end{aligned}$$

- (a) State what it means that a dynamical system is Hamiltonian and show that this system is Hamiltonian. [2]
  - (b) Find all equilibrium points for the system. [3]
  - (c) By linearisation, determine the nature of each equilibrium point. [4]
  - (d) Prove that  $H$  is conserved along trajectories. Show that the two saddle points have the same energy  $E_*$  and write the level set  $H(x, y) = E_*$  explicitly. Explain why this level set contains the separatrix through the saddles. [4]
  - (e) Sketch the phase portrait, indicating the saddles, centers, and the separatrix curve. [2]
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6. A two-dimensional dynamical system is given by

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= x(1 - x^4).\end{aligned}$$

- (a) Solve the system implicitly to find the equations of the phase paths  $y = y(x)$ . [3]
  - (b) Find all equilibria and classify them using linearisation. [3]
  - (c) State the Stable Manifold Theorem for a generic planar system near a hyperbolic fixed point. [2]
  - (d) Apply the Stable Manifold Theorem at  $(0, 0)$  and *verify it explicitly* by finding tangent vectors from the nonlinear phase-curve equations and showing that the dimension statement holds. [3]
  - (e) Sketch the phase portrait and describe the qualitative behaviour near each equilibrium. [4]
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7. This question concerns the damped pendulum,

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -\sin x - \gamma y,$$

for some constant  $\gamma \geq 0$ .

- (a) For  $\gamma = 0$ , find a first integral of the system. [3]
  - (b) Prove that the system cannot have closed orbits if  $\gamma > 0$ , citing any results you use (no need to prove them). [4]
  - (c) Define what is meant by a *Lyapunov function* of a dynamical system  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ . [3]
  - (d) For small  $\gamma > 0$ , find a Lyapunov function for this system and use it to prove that, except for two orbits ending at the top point  $(\pm\pi, 0)$ , all solutions tend to the origin. [5]
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8. Consider the following system

$$\dot{x} = -y + x^3 - x^5 \quad \text{and} \quad \dot{y} = x + y^3 - y^5.$$

The following identities may be useful:  $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta = \cos 2\theta$ ,  $\sin^4 \theta + \cos^4 \theta = \frac{1}{4}(3 + \cos 4\theta)$  and  $\sin^6 \theta + \cos^6 \theta = \frac{1}{8}(5 + 3 \cos 4\theta)$ .

- (a) Rewrite the system in polar coordinates. [3]
  - (b) Define what is meant by a *positively invariant set*. [2]
  - (c) For the system above, find  $R_1 > 0$  and  $R_2 < \infty$  such that the annulus  $\{R_1 \leq r \leq R_2\}$  is positively invariant. [4]
  - (d) State (do not prove) Poincaré–Bendixson Theorem. [3]
  - (e) Assuming that the above system has a single fixed point, what can you say about its possible  $\omega$ -limit sets? [3]
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