MATH3101/MATH31620 (Fluid Mechanics III/V)
YEAR 2025–26

FORMULA SHEET

Some vector identities:

$$\nabla \cdot (f\boldsymbol{A}) = (\nabla f) \cdot \boldsymbol{A} + f\nabla \cdot \boldsymbol{A} \tag{1}$$

$$\nabla \times (f\boldsymbol{A}) = (\nabla f) \times \boldsymbol{A} + f\nabla \times \boldsymbol{A} \tag{2}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{3}$$

$$\boldsymbol{A} \cdot \nabla \boldsymbol{A} = \tfrac{1}{2}\nabla|\boldsymbol{A}|^2 - \boldsymbol{A} \times (\nabla \times \boldsymbol{A}) \tag{4}$$

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \tag{5}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \boldsymbol{B} + \boldsymbol{A}\nabla \cdot \boldsymbol{B} - \boldsymbol{B}\nabla \cdot \boldsymbol{A} \tag{6}$$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \nabla \boldsymbol{B} + \boldsymbol{B} \cdot \nabla \boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \tag{7}$$

In cylindrical coordinates (r, θ, z):

$$\nabla f = \frac{\partial f}{\partial r}\widehat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\boldsymbol{e}}_\theta + \frac{\partial f}{\partial z}\widehat{\boldsymbol{e}}_z \tag{8}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{9}$$

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\widehat{\boldsymbol{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\widehat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\widehat{\boldsymbol{e}}_z \tag{10}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{11}$$

$$\nabla^2 \boldsymbol{A} = \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\widehat{\boldsymbol{e}}_r + \left(\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2}\right)\widehat{\boldsymbol{e}}_\theta + (\nabla^2 A_z)\widehat{\boldsymbol{e}}_z \tag{12}$$

$$\boldsymbol{B} \cdot \nabla \boldsymbol{A} = \left(\boldsymbol{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r}\right)\widehat{\boldsymbol{e}}_r + \left(\boldsymbol{B} \cdot \nabla A_\theta + \frac{B_\theta A_r}{r}\right)\widehat{\boldsymbol{e}}_\theta + (\boldsymbol{B} \cdot \nabla A_z)\,\widehat{\boldsymbol{e}}_z \tag{13}$$

In spherical coordinates (r, θ, ϕ):

$$\nabla f = \frac{\partial f}{\partial r}\widehat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\widehat{\boldsymbol{e}}_\phi \tag{14}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{15}$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\widehat{\boldsymbol{e}}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\widehat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\widehat{\boldsymbol{e}}_\phi \tag{16}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{17}$$

$$\begin{aligned}\nabla^2 \boldsymbol{A} = &\left(\nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{\partial A_\phi}{\partial \phi}\right]\right)\widehat{\boldsymbol{e}}_r \\ &+ \left(\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial \phi}\right)\widehat{\boldsymbol{e}}_\theta \\ &+ \left(\nabla^2 A_\phi + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \phi} - \frac{A_\phi}{r^2\sin^2\theta}\right)\widehat{\boldsymbol{e}}_\phi\end{aligned} \tag{18}$$

$$\begin{aligned}\boldsymbol{B} \cdot \nabla \boldsymbol{A} = &\left(\boldsymbol{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r}\right)\widehat{\boldsymbol{e}}_r + \left(\boldsymbol{B} \cdot \nabla A_\theta - \frac{B_\phi A_\phi}{r}\cot\theta + \frac{B_\theta A_r}{r}\right)\widehat{\boldsymbol{e}}_\theta \\ &+ \left(\boldsymbol{B} \cdot \nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r}\cot\theta\right)\widehat{\boldsymbol{e}}_\phi\end{aligned} \tag{19}$$

Bessel functions: $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + ru' + \left(r^2 - n^2\right)u = 0. \tag{20}$$

Both $J_n(r)$ and $Y_n(r) \to 0$ as $r \to \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \to \infty$ as $r \to 0$.