EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3101-WE01

Title:
Fluid Mechanics III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. A fluid moves two-dimensionally so that its velocity, $\boldsymbol{u}$, is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = x\widehat{\boldsymbol{e}}_x + (x+t)\widehat{\boldsymbol{e}}_y,$$

where $\widehat{\boldsymbol{e}}_x$ and $\widehat{\boldsymbol{e}}_y$ are the unit vectors for the Cartesian coordinates (x, y).

(a) Find the equation for the particle path of a particle released from $\boldsymbol{x} = (a, b)$ at $t = 0$. **[4]**

(b) Find the equation for a streamline passing through $\boldsymbol{x} = (x_0, y_0)$ at $t = 0$. **[4]**

(c) Is this flow compressible? Justify your answer. **[2]**

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2. The linearised equations for small-amplitude water waves on the surface of water with a mean depth h may be written in the form

$$\begin{aligned} \nabla^2\phi &= 0 \qquad \text{for } -h < z < 0, \\ \frac{\partial \phi}{\partial z} &= 0 \qquad \text{for } z = -h, \\ \frac{\partial \phi}{\partial t} + g\eta &= 0 \qquad \text{for } z = 0, \\ \frac{\partial \phi}{\partial z} &= \frac{\partial \eta}{\partial t} \qquad \text{for } z = 0, \end{aligned}$$

where $\phi(x, z, t)$ is the velocity potential, $\eta(x, t)$ is the free surface height, and g is the acceleration due to gravity.

(a) Assume a travelling wave solution of the form $\phi(x, z, t) = X(x - ct)Z(z)$ and find the functions $X(x - ct)$ and $Z(z)$. **[4]**

(b) Hence find the dispersion relation $\omega(k)$ for these waves. **[4]**

(c) Calculate c in the shallow water limit. In this limit are the waves dispersive? **[2]**

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3. (a) State the conditions under which a fluid flow $\boldsymbol{u}(\boldsymbol{x}, t)$ may be modelled by Burgers' equation of the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

[2]

(b) A fluid flow satisfies this equation with an initial condition $u_0(x)$. Show that along characteristics, u is constant and hence obtain a condition for when/if a smooth solution ceases to exist. **[4]**

(c) Apply your result to the initial condition

$$u_0(x) = A\cos(kx),$$

where A and k are positive constants, and hence determine whether a smooth solution exists for all $t > 0$. **[4]**

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4. (a) Starting with the Navier–Stokes equations, show that an incompressible, viscous, unforced fluid with velocity $\boldsymbol{u}$, vorticity $\boldsymbol{\omega}$ and kinematic viscosity ν satisfies

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{\omega}.$$

[5]

(b) For a two-dimensional flow in Cartesian coordinates, $\boldsymbol{u} = (u, v)$, *you are given* that this equation reduces to

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega. \qquad (4.1)$$

Now consider the stream function

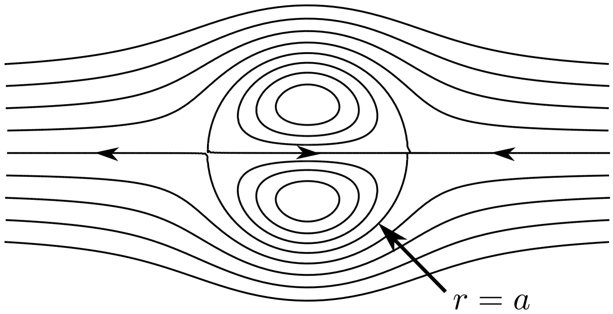
$$\psi = C\mathrm{e}^{-\lambda t}\cos(\alpha x)\cos(\beta y).$$

(i) Work out the velocity, (u, v), and hence the vorticity, ω, from this stream function. Write ω in terms of ψ. **[2]**

(ii) Using these results and eq. (4.1), determine the constant parameter λ in terms of α and β such that this stream function provides an exact solution of the Navier–Stokes equations for an arbitrary constant C. **[3]**

SECTION B

5.



Consider the 2D double vortex flow shown above. The double vortex is contained within $r \leq a$ and has non-zero vorticity. Outside $r = a$ the flow is potential. Working in polar coordinates, we will represent this flow in terms of a piecewise stream function

$$\psi(r,\theta) = \begin{cases} \psi_1(r,\theta) & r \leq a \\ \psi_2(r,\theta) & r > a, \end{cases}$$

such that $\boldsymbol{u} = (u_r, u_\theta) = \boldsymbol{\nabla} \times [\psi(r,\theta)\widehat{\boldsymbol{e}}_z]$. The flow is continuous and non-singular everywhere and becomes uniform at large distances.

(a) In the the region where $r > a$, show that $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$ implies that

$$\nabla^2\psi_2 = 0,$$

and construct a solution for ψ_2 satisfying $u_r(a,\theta) = 0$ and $\boldsymbol{u}(|\boldsymbol{x}| \to \infty) = -U\widehat{\boldsymbol{e}}_x$, where U is constant. You may use that the flux function for a dipole (that satisfies Laplace's equation) is given by $\psi_d = m\sin(\theta)/r$, where m is a constant. **[5]**

(b) In the region where $r < a$ it can be shown that $\nabla^2\psi_1 = -k^2\psi_1$, where k is a constant. Assuming that $\psi_1 = \sin(\theta)R(r)$, show that

$$R(r) = c_1 J_1(kr),$$

where c_1 is a constant and J_1 is a Bessel function of the first kind. **[4]**

(c) Explain why in order to enforce $u_r(a,\theta) = 0$ and to match the sketch above we should choose that $k = j_1/a$, where j_1 is the radius of the *first*, positive, non-zero root of the J_1 Bessel function. **[2]**

(d) Complete the solution by matching u_θ from both stream functions at $r = a$ to give c_1 as function of a, U and j_1. You may use that

$$\frac{\mathrm{d}}{\mathrm{d}x}[J_1(x)] = J_0(x) - \frac{J_1(x)}{x}.$$ **[4]**

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6. A simple model for a tornado is a Rankine vortex, with a flow velocity given by

$$\boldsymbol{u}(r,\theta,z) = \begin{cases} \dfrac{kr}{\ell^2}\widehat{\boldsymbol{e}}_\theta & r \le \ell \\ \dfrac{k}{r}\widehat{\boldsymbol{e}}_\theta & r > \ell, \end{cases}$$

where ℓ is the radius of the inner region and k is the strength of the vortex. We will assume that our tornado is described by the steady Euler equations and the only force is gravity.

(a) In the region $r > \ell$, the flow is steady and potential. Show that under these conditions the function

$$H := \frac{|\boldsymbol{u}|^2}{2} + \frac{p}{\rho_0} + gz$$

is constant *everywhere* in this region. **[3]**

(b) Using H, solve to find the pressure, p, in this region, assuming $p = p_0$ when $z = 0$ and $r \to \infty$. **[2]**

(c) Now consider the region $r < \ell$. Starting from the steady momentum equation, solve for pressure in this region, being careful to match the pressure at $r = \ell$. **[4]**

(d) The drop in pressure and temperature within the tornado takes the air below the dew point, creating water vapour and making the funnel visible. Let's assume this occurs when $p = p_0/2$. Use your solution for p from part (c) to find the surface $Z(r)$ that describes the shape of the funnel and sketch $Z(r)$. **[4]**

(e) Adam watches in horror as this tornado forms above his new house, with the top of his roof situated at $z = -h$. He sees that the tornado is slowly becoming more intense, such that it evolves through a series of steady states with progressively smaller ℓ. Find an expression for the value of ℓ at which Adam sees the tornado touch down on his roof, i.e. when Z first reaches $z = -h$. **[2]**

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7. Peter is listening to music in his office, which we model as the 2D rectangular domain

$$\{(x,y) : 0 < x < L,\ 0 < y < D\}.$$

We consider linear sound waves inside the office, described by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ where

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi.$$

(a) Initially all four walls of the office are rigid and no doors are open. Write down the appropriate boundary conditions for ϕ on all four boundaries. **[2]**

(b) By assuming a solution of the form

$$\phi(x,y,t) = X(x)Y(y)\mathrm{e}^{-\mathrm{i}\omega t},$$

show that X and Y satisfy the equations

$$X'' = -k_x^2 X, \quad Y'' = -k_y^2 Y,$$

and find k_y as a function of k_x. Hence find the possible values of the frequency, ω. **[5]**

(c) Peter is unsatisfied with the sound quality in his office and decides to add acoustic panelling to three of the walls: $x = 0$, $x = L$ and $y = D$. The spongy panelling material forces the pressure to be zero on these walls, but it allows the flow to have nonzero velocity. Find the possible values of ω in Peter's updated office. **[5]**

(d) Assuming $L > D$, compare the cutoff frequency, $\omega/2\pi$, in both cases. Peter claims that, after applying the acoustic material, he doesn't notice much difference in the sound. Comment, therefore, on the type of music it is likely that Peter is listening to. **[3]**

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8. Consider a steady flow of a Newtonian viscous fluid in an infinite pipe aligned with the z-axis. The pipe has an annular cross-section $a \leq r \leq b$. The fluid, with viscosity μ, flows between stationary boundaries at $r = a > 0$ and $r = b$ and there is no external body force.

(a) Starting from the dimensional incompressible Navier–Stokes equations, introduce a characteristic velocity U, length L, and pressure scale $P = \mu U/L$, and derive the nondimensional equations. Identify the Reynolds number. **[4]**

(b) From now on, all variables are nondimensional. Suppose the flow is steady and unidirectional, with velocity field

$$\boldsymbol{u} = u(r)\widehat{\boldsymbol{e}}_z.$$

Show that the governing equations reduce to the (nondimensionalised) Stokes equations,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, \\ \nabla^2 \boldsymbol{u} &= \boldsymbol{\nabla} p. \end{aligned}$$

By an appropriate choice of L, show that the nondimensional radial coordinate satisfies $c \leq r \leq 1$, where $c = a/b$. **[3]**

(c) Suppose a constant pressure gradient is applied, $\boldsymbol{\nabla} p = -\widehat{\boldsymbol{e}}_z$. Solve the governing equations subject to any boundary conditions and hence find the velocity profile $u(r)$. **[4]**

(d) Comment on the limit $c \to 0$ in the cases (i) $r = \mathcal{O}(1)$, and (ii) $r = \mathcal{O}(c)$. Hence, or otherwise, sketch the velocity profile for $0 < c \ll 1$. **[4]**
