Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2026 | MATH3111-WE01 |

| Title: |
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| Quantum Mechanics III |

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| Time: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.<br>Write your answer in the white-covered answer booklet with barcodes.<br>Begin your answer to each question on a new page. |

| Revision: | |
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ED01/2026

## SECTION A

**1.** (a) A quantum system has Hamiltonian $\hat{H}$ with discrete, non-degenerate eigenstates $\{|E_n\rangle\}$ and corresponding eigenvalues $E_n$. At time $t = 0$ the system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|E_1\rangle + |E_2\rangle\big).$$

Write down the state $|\psi(t)\rangle$ at a later time $t$. **[2]**

(b) Show that the probability of measuring the energy $E_1$ is independent of time. **[2]**

(c) Let $\hat{A}$ be an operator satisfying $[\hat{A}, \hat{H}] = 0$. Show directly (i.e. *without* using Ehrenfest's theorem) that the expectation value $\langle \hat{A} \rangle$ is time-independent for this state. **[4]**

(d) Briefly explain why the result in part (c) is physically reasonable. **[2]**

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**2.** (a) Let $\hat{\Omega}$ be a Hermitian operator with a discrete spectrum. State the spectral representation of $\hat{\Omega}$ in terms of its eigenvalues $\omega_n$ and projection operators, $\hat{P}_n$. **[2]**

(b) If the eigenvalues are non-degenerate write down the corresponding projection operator $\hat{P}_n$ on to the eigenstate $|\omega_n\rangle$. **[2]**

(c) Use your answer to show that the projection operators satisfy

$$\hat{P}_n \hat{P}_m = \delta_{nm} \hat{P}_n,$$

being careful to state clearly which properties of the eigenstates are required in your derivation. **[2]**

(d) Hence derive the spectral representation of the operator $\hat{\Omega}$. **[2]**

(e) Briefly explain the physical meaning of the projection operator $\hat{P}_n$ in the context of a measurement of $\Omega$. **[2]**

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**3.** A quantum particle is in a one-dimensional potential $V(x)$.

(a) In a region where $E > V(x)$, state the assumptions under which the WKB approximation is valid, and derive the leading-order WKB form of the wavefunction in this classically allowed region. **[3]**

(b) A particle is confined in an infinite square well $0 < x < L$ (so $\psi(0) = \psi(L) = 0$), but inside the well it experiences the piecewise potential

$$V(x) = \begin{cases} 0, & 0 < x < \frac{L}{3}, \\ V_0\left(\frac{3x}{L} - 1\right), & \frac{L}{3} < x < \frac{2L}{3}, \\ V_0\left(\frac{3x}{L} - 2\right)^2, & \frac{2L}{3} < x < L, \end{cases} \qquad (V_0 > 0).$$

Assume that $E > V_0$ and use the WKB approximation to derive a quantisation condition for the bound-state energies. Hence obtain an approximate expression for the ground-state energy $E_1$ in the limit $V_0 \ll E_1$.

You may use the result

$$\int \sqrt{a - bx^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a - bx^2} + \frac{a}{2\sqrt{b}}\sin^{-1}\left(x\sqrt{\frac{b}{a}}\right) + C.$$ **[7]**

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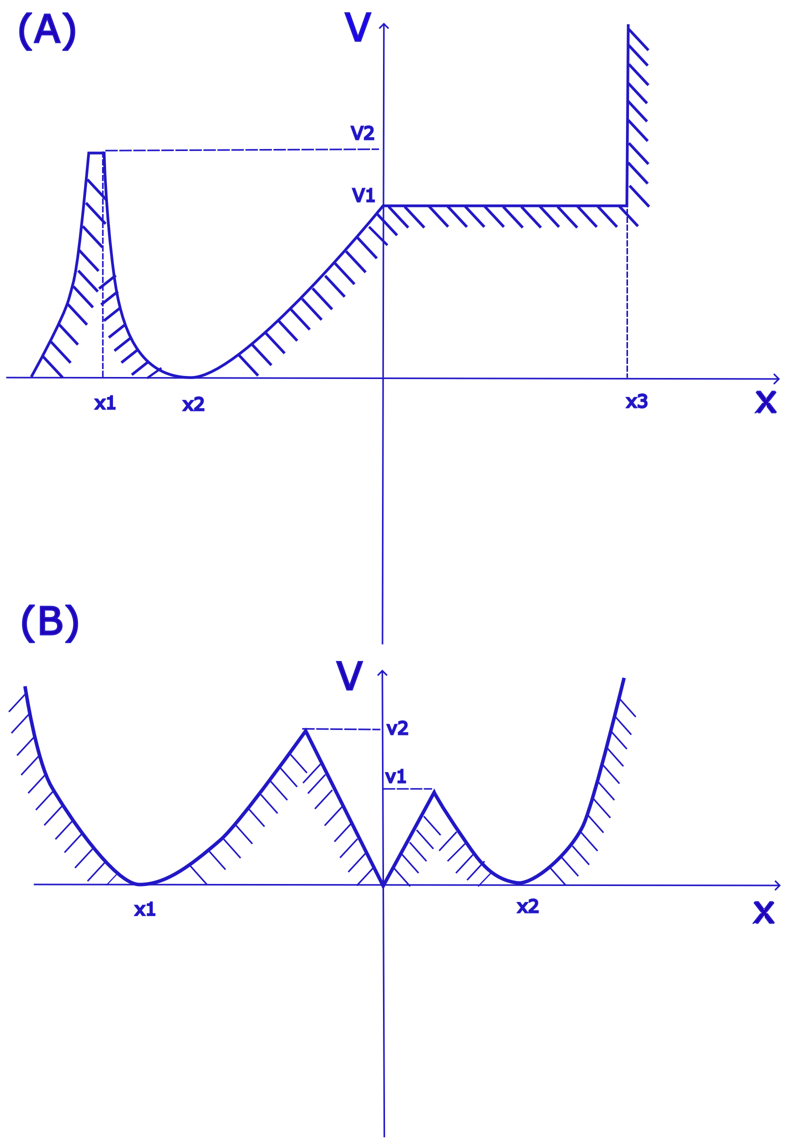
**4.** Figure shows two one dimensional quantum potentials.




In the potential (A) as $x \to \infty$, $V(x) \to +\infty$, and as $x \to -\infty$, $V(x) \to 0$ and in the potential (B) as $x \to \pm\infty$, $V(x) \to +\infty$.

(a) For each potential, describe the expected structure of the energy spectrum (discrete/continuous, bound/scattering states). **[5]**

(b) For each potential, sketch the qualitative form of the wave function for a particle of energy $E$, considering separately the cases $E < V_1$, $V_1 < E < V_2$ and $E > V_2$. Explain where the wavefunction is oscillatory and where it is exponentially decaying and define the classical turning points. **[5]**

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## SECTION B

**5.** A spin-$\frac{1}{2}$ particle with magnetic moment $\boldsymbol{\mu} = \gamma\hat{\mathbf{S}}$ is placed in a uniform magnetic field

$$\mathbf{B} = B_0\,\hat{\mathbf{z}},$$

such that the Hamiltonian $\hat{H}$ in terms of the Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ becomes

$$\hat{H} = -\frac{\hbar\gamma}{2}B_0\hat{\sigma}_z.$$

(a) The particle is initially prepared in the state $|\psi(0)\rangle = |+\rangle_x$, an eigenstate of $\hat{\sigma}_x$ with eigenvalue $+1$. Write this state in the $\hat{\sigma}_z$ eigenbasis. **[2]**

(b) Find the state $|\psi(t)\rangle$ at time $t$ and show that the expectation values $\langle\hat{\sigma}_x\rangle$, $\langle\hat{\sigma}_y\rangle$ and $\langle\hat{\sigma}_z\rangle$ describe Larmor precession about the $z$-axis. **[5]**

(c) Now suppose that, in addition to the field $B_0\hat{\mathbf{z}}$, a weak constant magnetic field $B_1\hat{\mathbf{x}}$ is applied, so that the Hamiltonian becomes

$$\hat{H} = -\frac{\hbar\gamma}{2}\big(B_0\hat{\sigma}_z + B_1\hat{\sigma}_x\big).$$

Without solving the time-dependent Schrödinger equation explicitly, determine whether $\langle\hat{\sigma}_z\rangle$ is time-independent. Justify your answer carefully. **[4]**

(d) Explain qualitatively how the motion of the spin expectation value differs from simple Larmor precession in this case. **[4]**

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**6.** (a) The Hamiltonian of the one-dimensional simple harmonic oscillator is

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

and the annihilation/creation operators are defined by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}.$$

Using $[\hat{x}, \hat{p}] = i\hbar$, show that

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

**[2]**

(b) Hence show that the Hamiltonian may be written in the form

$$\hat{H}_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

**[2]**

(c) Now consider adding a constant uniform force applied in the $x$-direction, so that the Hamiltonian becomes

$$\hat{H}_F = \hat{H}_0 - F\hat{x},$$

where $F$ is a real constant. Define a shifted annihilation operator

$$\hat{b} = \hat{a} - \alpha,$$

where $\alpha$ is a constant to be chosen. Show that $[\hat{b}, \hat{b}^\dagger] = 1$ and choose $\alpha$ so that $\hat{H}_F$ may be written as

$$\hat{H}_F = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + E_{\text{shift}},$$

for some constant $E_{\text{shift}}$, which you should determine. **[4]**

(d) Let $|0\rangle$ denote the ground state of $\hat{H}_0$ (so that $\hat{a}|0\rangle = 0$), and let $|0_F\rangle$ denote the ground state of $\hat{H}_F$. Show that $|0_F\rangle$ satisfies

$$\hat{a}|0_F\rangle = \alpha\,|0_F\rangle,$$

and say what kind of state $|0_F\rangle$ represents. **[3]**

(e) Recall that one of the outcomes of Ehrenfest's theorem is that the expectation values of the quantum system match those of the equivalent classical system. Determine $\langle\hat{x}\rangle$ for $|0_F\rangle$ and demonstrate that this is the case. **[4]**

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**7.** A particle of mass $m$ moves in three dimensions in the potential

$$V(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{2} m\omega^2 \big(\hat{x}^2 + \hat{y}^2 + \hat{z}^2\big),$$

with Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + V(\hat{x}, \hat{y}, \hat{z}).$$

Angular momentum operators $\hat{L}_i$ are represented by $\hat{L}_i = \epsilon_{ijk} x_j p_k$ for $i \in (x, y, z)$ and $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

(a) Show that $[\hat{L}_i, \hat{H}] = 0$ and $[\hat{L}^2, \hat{H}] = 0$. You may assume that $[\hat{p}^2, \hat{L}_i] = 0$ as shown in the lectures. State one physical implication of these results. **[3]**

(b) Separate the Schrödinger equation into radial and angular equations using the ansatz $\psi(r, \theta, \phi) = R(r) Y(\theta, \phi)$. You may use the Laplacian in spherical polar coordinates

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}.$$ **[3]**

(c) State the general solutions of the angular equation, introduce the quantum numbers, and explain what this implies about the degeneracy of the energy. **[3]**

(d) Introduce $u(r) = rR(r)$ and rewrite the radial equation in terms of $u(r)$. Solve the resulting equation for the ground state using the ansatz

$$u(r) = B\, r\, e^{-\alpha r^2/2}.$$

Work out $\alpha$ and the ground-state energy $E_0$, and state the degeneracy of the ground state. **[4]**

(e) Write the effective potential $V_{\text{eff}}(r)$ appearing in the radial equation. Explain what new quantum number this introduces compared with the one-dimensional harmonic oscillator. Hence state the minimal guaranteed degeneracy of an energy level, and briefly justify your answer. State which quantum numbers determine the energy of the system, and explain whether the minimal guaranteed degeneracy must always equal the total degeneracy. **[2]**

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**8.** An isotropic $N$-dimensional harmonic oscillator ($N > 4$) has Hamiltonian

$$\hat{H}_0 = \sum_{k=1}^{N} \left( \frac{\hat{p}_k^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_k^2 \right).$$

A small perturbation

$$\hat{H}' = \varepsilon\, m\omega^2\, (\hat{x}_1)^2 (\hat{x}_2)^2, \qquad |\varepsilon| \ll 1,$$

is added to the Hamiltonian $\hat{H}_0$ so that the total Hamiltonian is

$$\hat{H} = \hat{H}_0 + \hat{H}'.$$

(a) Work out the energy spectrum of the unperturbed system $E_n^{(0)}$, where $n = 0, 1, 2, \ldots$. State the degeneracies of the energy levels $n = 0$, $n = 1$ and $n = 2$. **[3]**

(b) Starting from the fact that $\hat{x}_k \sim \hat{a}_k + \hat{a}_k^\dagger$ (where $\hat{a}_k, \hat{a}_k^\dagger$ are ladder operators for each coordinate), investigate which states can produce nonzero contributions to the matrix elements of $\hat{H}'$. **[3]**

(c) Focus on the states in the first excited level $n = 1$. Using an appropriate basis, construct the matrix of $\hat{H}'$ restricted to this degenerate subspace and diagonalise it. Hence find the corrected energies at first order in $\epsilon$ and the corresponding (normalised) perturbed eigenstates in this subspace. **[4]**

(d) Focus now on the states in the second excited level $n = 2$. Identify the states in this space which are affected by $\hat{H}'$ at first order, and compute the first-order energy shifts. Your final answer should include (a) the shifts for the coupled states and (b) the degeneracy of the unshifted states at first order in $\varepsilon$. **[5]**

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