Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH31220-WE01

Title: Geometry of Mathematical Physics V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider the group $G = U(2)$ with respect to matrix multiplication. In each of the following cases, explain whether the given map defines a representation of G, and if so, whether the representation is faithful and irreducible.

(a) Let $V = \mathbb{C}$ and define

$$r_1 : G \to GL(V)\,,$$
$$r_1(g) = \det(g)\,.$$

[5]

(b) Let $V = \mathbb{C}^4$ and define

$$r_2 : G \to GL(V)\,,$$
$$r_2(g) = \begin{pmatrix} g & \mathbb{1}_2 \\ \mathbf{0} & g \end{pmatrix}\,.$$

Here $\mathbb{1}_2$ denotes the 2×2 identity matrix while $\mathbf{0}$ denotes the 2×2 zero matrix. [5]

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2. Let us define

$$H = \left\{ \begin{pmatrix} g_1 & \mathbf{0} \\ \mathbf{0} & g_2 \end{pmatrix} \mid g_1, g_2 \in SU(2) \right\}\,,$$

where $\mathbf{0}$ denotes the 2×2 zero matrix.

(a) Show that H is a subgroup of $SU(4)$. [5]

(b) Consider the fundamental representation of $SU(4)$

$$R : SU(4) \to GL(4, \mathbb{C})\,, \qquad R(U) = U\,, \quad \forall U \in SU(4)\,.$$

Show that by restricting R to H we obtain a representation of H. Determine whether or not this representation of H is irreducible. [5]

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3. The field strength tensor of electromagnetism is given in terms of electric and magnetic fields by

$$F^{\mu\nu} := \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} .$$

Write the following expressions in terms of electric and magnetic fields:

(a) $F_{\mu\nu}$ **[2]**

(b) $F^{\mu\nu}F_{\mu\nu}$ **[4]**

(c) $F^{\mu\nu}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$ **[4]**

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4. Let Φ transform under the gauge group $U(1)$ as

$$\Phi \to e^{in\theta(\mathbf{x})}\Phi$$

for $n \in \mathbb{Z}$.

(a) Work out the transformation of $\partial_\mu \Phi$. **[3]**

(b) Write down the covariant derivative $D_\mu \Phi$ and show that it has the correct transformation behaviour. **[3]**

(c) Consider the following action for Φ:

$$S = -\int d^4x \left[D_\mu \Phi \overline{D^\mu \Phi} + U(|\Phi|^2) + \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right] .$$

Find the conserved current using Noether's theorem. **[4]**

SECTION B

5. Let us define

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} .$$

(a) Show that G forms a group with respect to matrix multiplication which you may assume is associative. **[5]**

(b) Find a basis for the Lie algebra $\mathfrak{g}$ of the group G and compute its structure constants. **[5]**

(c) Show that the exponential map

$$\exp : \mathfrak{g} \to G$$

is surjective, that is, show that every element of G can be written as e^{γ} for some $\gamma \in \mathfrak{g}$. **[5]**

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6. Let $\mathfrak{g} = \text{span}_{\mathbb{R}}\{\mathbf{x}, \mathbf{y}\}$ be an abstract two-dimensional real Lie algebra spanned by the basis elements $\mathbf{x}, \mathbf{y}$.

(a) Show that if $[\mathbf{x}, \mathbf{y}] = 0$ then $\mathfrak{g}$ must be Abelian, i.e.

$$[\mathbf{v}, \mathbf{w}] = 0 \,, \qquad \forall\, \mathbf{v}, \mathbf{w} \in \mathfrak{g} \,.$$

[4]

(b) Let us assume now that $[\mathbf{x}, \mathbf{y}] \neq \mathbf{0}$. Constuct a basis $\{\mathbf{a}, \mathbf{b}\}$ for $\mathfrak{g}$ such that

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \,.$$

[**Hint**: use $[\mathbf{x}, \mathbf{y}]$ itself to help construct the required basis for $\mathfrak{g}$.] **[6]**

(c) Let $\mathfrak{g} = \text{span}_{\mathbb{R}}\{\mathbf{a}, \mathbf{b}\}$ be the same Lie algebra and basis as in part (b). Consider the linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(2, \mathbb{R})$ defined on a general Lie algebra element $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} \in \mathfrak{g}$ as

$$\rho(\mathbf{v}) = \rho(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\rho(\mathbf{a}) + \beta\rho(\mathbf{b}) \,, \qquad \alpha, \beta \in \mathbb{R} \,,$$

where

$$\rho(\mathbf{a}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,, \qquad \rho(\mathbf{b}) = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,.$$

Show that ρ defines a Lie algebra representation for the Lie algebra $\mathfrak{g}$. **[5]**

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7. Let ϕ be a complex scalar field with Lagrangian

$$\mathcal{L} = \overline{\partial_\mu \phi}\, \partial^\mu \phi \,.$$

(a) State the transformation of a scalar field under Lorentz transformations and explain why the action above defines a Lorentz invariant field theory. [**4**]

(b) Find the equations of motion and show that they are solved by

$$\phi = e^{ik_\mu x^\mu} \,,$$

whenever $k_\mu k^\mu = 0$. [**4**]

(c) How does the equation of motion change if we include a term $m^2|\phi|^2$ in the Lagrangian? Under which condition is $\phi = e^{ik_\mu x^\mu}$ a solution now? [**2**]

(d) A general solution to the equations of motion for $m = 0$ is given by

$$\phi(\mathbf{x}) = \int d^4k \, \delta(k^\mu k_\mu)\, e^{ik_\mu x^\mu} f(\mathbf{k}) \,,$$

for an arbitrary smooth function $f(\mathbf{k})$. Use this to show that the field theory in question is Lorentz invariant. [**5**]

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8. Let us consider the following Lagrangian for a real scalar ϕ and three Dirac spinors Ψ_i:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{i=1}^{3}\left(\bar{\Psi}_i\gamma^\mu\partial_\mu\Psi_i + y_i\bar{\Psi}_i\Psi_i\phi\right) \, .$$

(a) Find the equations of motion. **[5]**

(b) Using the defining representation, we can act with the group $U(3)$ on the Ψ_i, i.e. for $U \in U(3)$ we let $\Psi_i \rightarrow U_{ij}\Psi_j$. In each of the following cases, decide which $U \in U(3)$ constitute global symmetries of $\mathcal{L}$:

(i) $y_1 = y_2 = y_3$,

(ii) $y_1 = y_2 \neq y_3$,

(iii) $y_i \neq y_j \ \forall i \neq j$. **[4]**

(c) Assume that the two further terms

$$\delta\mathcal{L} = \mathrm{tr}\left(\partial_\mu\Omega\partial^\mu\Omega\right) + \sum_{i,j=1}^{3}\bar{\Psi}_i\Omega_{ij}\Psi_j$$

are included in the action, and that the field Ω is a matrix (with components Ω_{ij}) which transforms in the adjoint representation of $U(3)$. How does this change the analysis you have done in part (b) ? **[3]**

(d) Let $\Omega = \sum_a \Omega_a t_a$ with t_a a basis of $\mathfrak{u}(3)$ such that $\mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$. Determine the range of a and write $\delta\mathcal{L}$ in terms of the components Ω_a. **[3]**
