EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May/June	2026	MATH3171-WE01

Title:
Mathematical Biology

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. It's going to be O-k

(a) Consider the following models for the population of a single species, $x(t)$, evolving over time, t. Each model depends on a parameter, $k > 0$. For each model, provide a short biological interpretation and state any permissible equilibria and their stabilities. It may help to draw a phase portrait of $\mathrm{d}x/\mathrm{d}t$ against x in each case.

(i) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = kx,$

(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - k,$

(iii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -x(x+k)(x-k).$ **[6]**

(b) Draw a bifurcation diagram of permissible equilibria, x_0, against k for model (ii) above, indicating stability appropriately on the diagram. **[4]**

---

2. Two-species interaction The growth of two interacting species, $x(t)$ and $y(t)$, over time, t, is modelled by the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x-y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(x-b),$$

for a parameter $b > 0$.

(a) Find the permissible equilibria of the system and classify them in terms of their stability and their shape in the phase plane. **[4]**

(b) Assuming $b = 1/2$, draw the phase plane of y against x for the region $x, y \geq 0$, including sample trajectories, clearly marking nullclines and equilibria. **[4]**

(c) What happens to solutions starting with either $x = 0$ or $y = 0$? What is the biological interpretation of this? **[2]**

---

3. **Boxing up the budworms** Consider a spatial version of the spruce budworm model,

$$\frac{\partial u}{\partial t} = D\nabla^2 u + r\left[u\left(1 - \frac{u}{2}\right) - \frac{u^2}{1+u^2}\right],$$

on the square domain $x \in [0, L]$, $y \in [0, L]$, for positive parameters D, r, L.

(a) Determine the spatially homogeneous equilibria, and their stability in the absence of diffusion. **[4]**

(b) Consider now homogeneous Dirichlet boundary conditions on the boundaries of the boxed domain. Using a linear stability analysis, show that the population extinction can be made stable for suitable parameters. Determine a critical domain size L for extinction to be stable. **[6]**

---

4. **Of spiders and exponentials** Consider the discrete-time model for a population u_n:

$$u_{n+1} = u_n \exp[r(1 - u_n)(u_n - A)],$$

where $r > 0$ and $A \in (0, 1)$.

(a) Find all permissible equilibria of this model, and analytically determine their stability. Note any parameters where stability of any equilibria changes within the ranges above. **[6]**

(b) Draw cobweb diagrams illustrating how trajectories in this model behave. Draw at least one diagram for each parameter region where the stability of any equilibrium changes. Briefly describe the biological interpretation of this model's behaviours, especially commenting on the effects of initial population sizes. **[4]**

SECTION B

5. **'Nothing beats a Jet2 holiday'** While staying in a hotel, you discover a population of cockroaches confined to a long, narrow corridor. The hotel management offer you free breakfast if you can solve the problem for them. You therefore propose a one-dimensional model for the cockroach density $u(x,t)$,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + v\frac{\partial u}{\partial x}, \quad u(x,0) = \begin{cases} 0 & x < 0 \\ a & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}, \quad u(\pm\infty, t) = 0, \qquad (5.1)$$

where a, D and v are positive constants and $x \in \mathbb{R}$.

(a) Interpret the terms in this setup in the context of the cockroaches and their environment. What sort of behaviour do you expect to see from this model? **[3]**

(b) Consider the total number of cockroaches,

$$M = \int_{-\infty}^{\infty} u \, \mathrm{d}x.$$

In addition to $u \to 0$ as $x \to \pm\infty$, what must be true about the behaviour of the solution at infinity in order for M to be conserved in time? **[3]**

(c) Using Fourier transforms, solve the fundamental problem for (5.1) and then solve the full problem. You may find the following useful:

$$\mathcal{F}[f(x)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x) \, \mathrm{d}x,$$

$$\mathcal{F}^{-1}[g(k)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} g(k) \, \mathrm{d}k,$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+bx+c} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \mathrm{e}^{c} \mathrm{e}^{b^2/(4a)},$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} \mathrm{e}^{-t^2} \mathrm{d}t.$$

[6]

(d) Suppose now that v is measured to be small enough that it can be assumed to be zero. At $x = 10$ the corridor ends in a wall and when the cockroaches reach the wall, they turn around and head back. If you model the corridor as a semi-infinite line, $x \in (-\infty, 10]$, will the approach used in this question still work? If so, say how it can be adapted for the new situation. If not, explain why not. **[3]**

---

6. **Travelling squirrels with non-constant diffusion** Consider a model for the growth and spread of a population of red squirrels in a long woodland corridor,

$$\frac{\partial u}{\partial t} = D\frac{\partial}{\partial x}\left[\left(1+\frac{u}{4K}\right)\frac{\partial u}{\partial x}\right] + au\left(1-\frac{u}{K}\right),$$

where $u(x,t)$ is the density of the population at position $x \in \mathbb{R}$ and time t and where D, K and a are positive parameters.

(a) Give a brief biological interpretation of the terms in the equation. **[2]**

(b) By writing

$$u = U\widehat{u}, \quad x = X\widehat{x}, \quad t = T\widehat{t},$$

where variables with hats are dimensionless, show that the model can be written in dimensionless form as, once hats are removed,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(4+u)\frac{\partial u}{\partial x}\right] + u(1-u). \tag{6.1}$$

You should specify the values of U, X and T you choose in terms of the original problem's parameters. **[3]**

(c) We are now going to look for a travelling wave solution, $u(z)$, where $z = x - ct$ and $c > 0$ is the wavespeed. State the stability of the equilibria of the homogeneous form of (6.1) and hence propose boundary conditions for $u(z)$ at $z = \pm\infty$. **[3]**

(d) Now write (6.1) for $u(z)$. Perform a linear stability analysis on the homogeneous equilibria to find the speed of the wave in the nondimensionalised system. **[5]**

(e) Finally, suppose the squirrels initially occupy the region $x < 0$. Estimate how long it will take for the population front to reach $x = L$ (in the dimensional system). **[2]**

---

7. The difference of one term for Turing

(a) In words, describe what is meant by a *Turing instability*. Why are these instabilities potentially important in biology? [**3**]

(b) For each of the following models of populations, give a brief explanation why they can or can not have Turing instabilities. Clearly state any conditions on functions or parameters necessary to generate a Turing instability for the spatial domain $x \in [0, L]$, where $L \gg 1$ is arbitrarily large (i.e. you can approximate the domain size L as infinite).

(i) $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + f(u)$,

(ii) $\dfrac{\partial u}{\partial t} = -a\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^4 u}{\partial x^4} + f(u)$. [**12**]

---

8. Taxis praxis Consider the following cross-diffusion model of two interacting populations:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + a\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + u(b - u - v),\\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + uv - v,\end{aligned}$$

on the interval $x \in [0, L]$ with homogeneous Neumann boundary conditions. Throughout, consider $b > 1$, and $a \in \mathbb{R}$.

(a) Determine the unique spatially homogeneous equilibrium which is suitable for a Turing instability analysis. [4]

(b) Linearise the full system of PDEs around this spatially homogeneous equilibrium. Determine the matrices $\boldsymbol{D}$ and $\boldsymbol{J}$ such that the linear system of perturbations $\boldsymbol{u}_1$ can be written in vector form as

$$\frac{\partial \boldsymbol{u}_1}{\partial t} = \boldsymbol{D}\frac{\partial^2 \boldsymbol{u}_1}{\partial x^2} + \boldsymbol{J}\boldsymbol{u}_1. \quad [4]$$

(c) Solve this linear system, and hence deduce necessary conditions on a and b for the system to admit Turing instabilities. You may assume that L is arbitrarily large, and you should use the ansatz $\boldsymbol{u}_1 \propto \mathrm{e}^{\lambda_k t} w_k(x)$, stating the appropriate eigenfunctions $w_k(x)$. Your result should be two relatively simple inequalities involving a and b. [7]
