EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3201-WE01

Title:
Geometry III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Is it true or false that:

(a) The group of isometries of the Euclidean plane can be generated by all glide reflections? Justify your answer. **[5]**

(b) The group of orientation-preserving isometries of the Euclidean plane can be generated by all translations? Justify your answer. **[5]**

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2. Let $A_1A_2A_3$ be a triangle in the Euclidean plane. Denote also $A_4 = A_1$. Let B_i, C_i, D_i, $i = 1, 2, 3$ be points on A_iA_{i+1} such that $A_iB_i = B_iC_i = C_iD_i = D_iA_{i+1}$.

(a) Let B_0 be the common point of the medians of $\triangle B_1B_2B_3$ and let C_0 be the common point of the medians of $\triangle C_1C_2C_3$. Show that $B_0 = C_0$. **[5]**

(b) Let $\widehat{B}$ be the common point of the altitudes of $\triangle B_1B_2B_3$ and let $\widehat{C}$ be the common point of the altitudes of $\triangle C_1C_2C_3$. Is it always true that $\widehat{B} = \widehat{C}$? **[5]**

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3. Let $A_1A_2A_3A_4A_5A_6$ be an ideal hexagon in the hyperbolic plane (i.e. a hexagon with all vertices at the boundary). Denote also $A_7 = A_1$. Assume that there exists a point O inside this hexagon such that $\angle A_iOA_{i+1} = \frac{\pi}{3}$ for $i = 1, \ldots, 6$.

(a) Show that there exists a circle inscribed in $A_1A_2A_3A_4A_5A_6$. **[5]**

(b) Let r be the radius of the circle inscribed in $A_1A_2A_3A_4A_5A_6$. Find $\cosh r$. **[5]**

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4. Let KLM be a triangle in the unit sphere. Suppose that $\angle KLM = \pi/2$ and let LH be the altitude through L. Denote $a = KH$ and $b = HM$, denote $h = LH$.

(a) Find $(\cos h)^2$ as a function of a and b. **[5]**

(b) Find the area of the triangle KLM, if $KM = \pi/2$. **[5]**

SECTION B

5. Let γ_0 be a circle in the Euclidean plane and let A, B, C, D be distinct points of γ_0 listed in cyclic order. Let γ_1 be the circle through A and C orthogonal to γ_0. Let γ_2 be the circle through B and D orthogonal to γ_0. Assume $[A, B, C, D] = 2$.

(a) Show that the circles γ_1 and γ_2 intersect and find the angle between them. **[5]**

(b) Let i_0, i_1, i_2 be inversions with respect to $\gamma_0, \gamma_1, \gamma_2$ respectively. Let G be the group generated by $i_0 i_1$ and $i_0 i_2$. Show that G is a finite subgroup of the group of Möbius transformations on $\mathbb{C} \cup \{\infty\}$. **[5]**

(c) Show that all the non-trivial elements of the group G defined in part (b) are elliptic. Find all possible orders of $g \in G$. **[5]**

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6. (a) Let A and B be points in the Euclidean plane, let a_1, a_2 be lines through A and $b_1, \dots, b_4$ be lines through B. Denote $S_{i,j} = a_i \cap b_j$, and consider the lines $p_i = S_{1,i}S_{2,i+1}$ for $i = 1, 2, 3$ and $q_i = S_{2,i}S_{1,i+1}$ for $i = 1, 2, 3$. Assume that none of the lines p_i and q_j are parallel to each other. Show that the lines p_1, p_2, p_3 are concurrent if and only if the lines q_1, q_2, q_3 are concurrent. **[5]**

(b) Formulate the statement dual to the one stated in part (a). **[5]**

(c) Is it true that the group of projective transformations acts transitively on the configurations described in part (a) and satisfying the condition that the lines p_1, p_2, p_3 are concurrent? Justify your answer. **[5]**

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7. Let h_1 and h_2 be two distinct horocycles centred at the same point of the boundary of the hyperbolic plane.

(a) Show that there exists a set of hyperbolic circles C_i, $i \in \mathbb{Z}$ such that all C_i are tangent to both h_1 and h_2 and C_i is tangent to C_{i+1} for all $i \in \mathbb{Z}$. **[5]**

(b) Let O_i be the (hyperbolic) centre of the circle C_i described in part (a), let $A_i = C_{i-1} \cap C_i$. Which of the (hyperbolic) segments is shorter, $O_i O_{i+1}$ or $A_i A_{i+1}$? Justify your answer. **[5]**

(c) In the above notation, assume that $\angle A_i O_i A_{i+1} = \pi/2$. Find the (hyperbolic) radius of C_i. **[5]**

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8. (a) Let H_1 and H_2 be two hyperbolic hexagons. Assume that each of H_1 and H_2 has six right angles. Is it always true that there exists an isometry of the hyperbolic plane taking H_1 to H_2? Justify your answer. **[5]**

(b) Suppose that $ABCDEF$ is a hyperbolic hexagon with six right angles, suppose also that the opposite sides of $ABCDEF$ are equal. Show that the diagonals AD, BE and CF have a common point O. **[5]**

(c) Let $ABCDEF$ be a right-angled hyperbolic hexagon with equal opposite sides and O as in part (b). Is it true or false that the area of $\triangle OAB$ is always equal to the area of $\triangle OBC$? Justify your answer. **[5]**
