Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3251-WE01

Title:

Stochastic Processes III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider a birth and death process $(X(t))_{t\geq 0}$ on $\mathcal{I} = \{0, 1, 2, \dots\}$ with birth rate $\lambda_n = \lambda$ and death rate $\mu_n = n\mu > 0$ in state $X(t) = n$ where $\lambda, \mu > 0$.

(a) Write down the Q matrix for this Markov process. **[3]**

(b) Let $\rho = \frac{\lambda}{\mu}$. Find the invariant distribution of this Markov Process in terms of ρ. **[7]**

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2. Suppose that minor defects over the railway tracks between Durham and York stations are distributed as a Poisson process with rate λ_1, and that, independently, major defects are distributed over the railway tracks according to a Poisson process with rate λ_2. Let $N(t)$ be the total number of defects in the first t miles of railway track from the station in Durham.

(a) If exactly 1 defect has been found in the first 10 miles of railway track from Durham, that is $N(10) = 1$, what is the probability it is within the first 2 miles of track from Durham? **[3]**

(b) If exactly 1 defect has been found in the first t miles of railway track from Durham, what is the probability that it is a minor defect? **[3]**

(c) What is the expected number of minor defects between two successive major defects? **[4]**

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3. In each of the following problems, $(Z_n)_{n\geq 0}$ is a branching process.

(a) Suppose the offspring distribution $(p_k)_{k\geq 0}$ has $p_0 = \frac{1}{3}$, $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{3}$, and $p_k = 0$ otherwise. Is $\lim_{n\to\infty} \mathbb{P}[Z_n = 0] = 0$? **[5]**

(b) Suppose the offspring distribution $(p_k)_{k\geq 0}$ has associated generating function $\phi(s) = s^3$. Is $\lim_{n\to\infty} \mathbb{P}[Z_n = 0] = 0$? **[5]**

For each problem you must fully justify your solution.

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4. Let X and Y be independent random variables representing fair six- and twelve-sided die, respectively. That is, $\mathbb{P}[X = j] = \frac{1}{6}$ for $j \in \{1, \dots, 6\}$ and $\mathbb{P}[Y = j] = \frac{1}{12}$ for $j \in \{1, \dots, 12\}$.

(a) Compute the total variation distance between the distributions of X and Y. **[6]**

(b) Construct a coupling $(\widetilde{X}, \widetilde{Y})$ of X and Y such that $\widetilde{\mathbb{P}}[\widetilde{X} \neq \widetilde{Y}] = \frac{1}{2}$. You should explain why your construction gives a valid coupling. **[4]**

For each problem you must fully justify your solution.

SECTION B

5. Let $X(t)$ be a continuous time Markov process on the state space $\mathcal{I} = \{1, 2, 3\}$ with generator (Q-matrix)

$$Q = \begin{pmatrix} -4 & 4 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

(a) Show that this Markov process is irreducible. **[3]**

(b) Find the characteristic polynomial of Q and identify the eigenvalues. **[3]**

(c) Compute $\mathbb{P}[X(t) = 3|X(0) = 1]$ and find $\lim_{t\to\infty} \mathbb{P}[X(t) = 3|X(0) = 1]$. **[4]**

(d) Let $T_2 = \inf\{t \geq 0 : X(t) = 2\}$ be the first time the Markov process is in state 2. Compute $\mathbb{P}[T_2 \leq t|X(0) = 1]$. **[5]**

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6. A standard fair die is tossed repeatedly. Let T be the number of tosses until the sequence 1-2-3-1-2-3 is observed for the first time. By exhibiting an appropriate martingale, find $\mathbb{E}[T]$.

You must fully justify your answer stating any results used from the course, including a full check of the hypothesis of any theorems used. **[15]**

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7. Let X be a non-negative discrete random variable with $0 < \mathbb{E}[X] < \infty$. Recall the *size-biased* distribution $\widehat{X}$ of X is defined by

$$\mathbb{P}[\widehat{X} = x] = \frac{\mathbb{E}[x 1_{X=x}]}{\mathbb{E}[X]}.$$

This problem uses $X \preceq Y$ to denote that X is stochastically dominated by Y, and $\mathrm{Poi}(\lambda)$ to denote the Poisson distribution with parameter $\lambda > 0$.

(a) Suppose Y is a $\mathrm{Poi}(\lambda)$ random variable. We showed in the lectures that $\widehat{Y} \overset{(d)}{=} Y + 1$. Using this fact, show that $Y \preceq \widehat{Y}$. **[3]**

(b) Suppose W is a random variable with finite mean, and that f, g are increasing functions. Prove Chebyshev's correlation inequality, which states that

$$\mathbb{E}[f(W)g(W)] \geq \mathbb{E}[f(W)]\mathbb{E}[g(W)].$$

Hint: let W_1, W_2 be independent and distributed as W, and consider $\mathbb{E}[(f(W_1) - f(W_2))(g(W_1) - g(W_2))]$. **[3]**

(c) Show, by using Chebyshev's correlation inequality or otherwise, that $X \preceq \widehat{X}$ for all non-negative discrete random variables X with $0 < \mathbb{E}[X] < \infty$. **[5]**

(d) Suppose that W is a non-negative integer-valued random variable with finite mean $\lambda \in (0, \infty)$, and that $\widehat{W} \overset{(d)}{=} W + 1$. Show that W is a $\mathrm{Poi}(\lambda)$ random variable. **[4]**

For each problem you must fully justify your solution.

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8. Fix $\lambda > 0$, and consider the recurrence $a_{n+2} = a_{n+1} + \lambda a_n$ for $n \geq 0$, with initial conditions $a_0 = 1$ and $a_1 = 1$.

(a) Show there exists a $\theta \in \mathbb{R}$ such that the sequence $u_n = \theta^n a_n$ satisfies a renewal equation. That is, find a function $u(t)$ such that $u(n) = u_n$, a random variable X, and a function $c(t)$ such that **[7]**

$$u(t) = c(t) + \mathbb{E}[u(t - X)].$$

(b) Use renewal theory to compute the exponential growth rate $\lim_{n\to\infty} \frac{1}{n} \log a_n$ of the sequence a_n. **[8]**

For each problem you must fully justify your answer, including a full check of the hypothesis of any theorems used.
