Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3281-WE01

Title:
Topology III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $X = \{1, 2, 3, 4\}$.

(a) Consider the following topologies on X.

$$\tau_a = \{\emptyset, \{1\}, \{1, 2\}, X\} \quad \tau_b = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}$$

$$\tau_c = \{\emptyset, \{1\}, \{1, 2, 3\}, X\} \quad \tau_d = \{\emptyset, \{2\}, \{2, 4\}, X\}$$

Which of the topological spaces $(X, \tau_a), (X, \tau_b), (X, \tau_c)$ and (X, τ_d) are homeomorphic? If a pair are homeomorphic, give a homeomorphism; if they are not, explain why not. **[3]**

For any set Y and finite collection of subsets $\tau_Y = \{\emptyset, U_1, U_2, \ldots, U_k, Y\}$, we say τ_Y is *nested* if $\emptyset \subset U_1 \subset U_2 \subset \cdots \subset U_k \subset Y$, where $k \geq 1$. (Note that these inclusions are strict.)

(b) Give two topologies on X, one nested and one not, each containing 5 sets. **[2]**

(c) Show in general that if a collection τ_Y is nested then it is a topology on Y. **[2]**

(d) What does it mean for a topological space to be Hausdorff? If τ_Y is nested then is (Y, τ_Y) always, sometimes, or never a Hausdorff space? Explain. **[3]**

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2. In $\mathbb{R}^n$ with the standard topology, we define the closed ball and the sphere:

$$D^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}, \quad S^{n-1} = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

(a) Show that both sets are compact. If you use a result from lectures, state it clearly. **[4]**

In a lecture, we proved that for $n \geq 1$ the quotient space

$$D^n / S^{n-1} \cong S^n \subseteq \mathbb{R}^{n+1}.$$

Prove this for $n = 1$, as follows:

(b) Draw simple sketches of the three sets involved.
Give explicitly a suitable continuous, surjective function $f : D^1 \to S^1$, and define a function $\overline{f} : D^1/S^0 \to S^1$ in terms of f. **[3]**

(c) Show that $\overline{f}$ is a homeomorphism. Explain clearly, stating any results from lectures that you use, but detailed calculations are not required. **[3]**

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3. (a) State what it means for two topological spaces to be homotopy equivalent. **[2]**

(b) Consider the lists of upper- and lower-case letters below (in the given font!).

F G H I J
f g h i j

Viewing each letter as a subset of $\mathbb{R}^2$ equipped with the subspace topology, partition the upper-case list, the lower-case list and the combined list, respectively, into sets of homotopy-equivalent topological spaces. In particular, identify any letters from the upper-case list which are not homotopy equivalent to their lower-case counterparts. Briefly justify your answers, including by making reference to appropriate topological invariants wherever necessary. **[4]**

(c) Let A be the annulus $A = \{z \in \mathbb{C} \mid 1 \leq |z| < 2\}$, let S^1 be the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and let X be an arbitrary non-empty compact topological space. Prove that the product $X \times A$ is homotopy equivalent, but not homeomorphic, to the product $X \times S^1$. **[4]**

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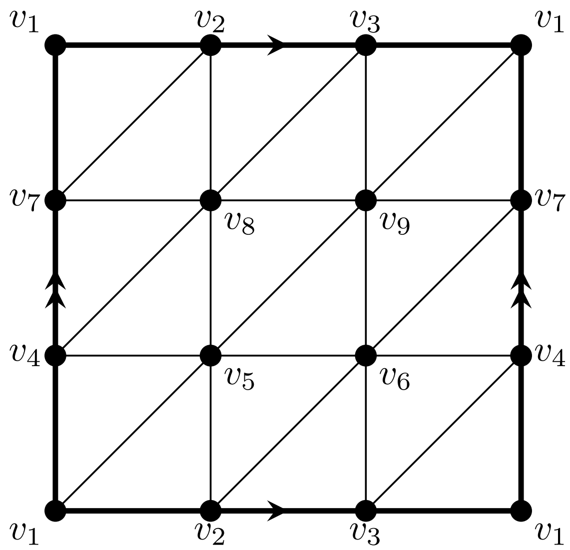
4. (a) If K and L are finite simplicial complexes, state what it means for a map $f : K \to L$ to be a simplicial map. **[2]**

(b) Let K be the 2-dimensional finite simplicial complex which triangulates the torus $T = S^1 \times S^1$ and is given (via the identifications indicated by the arrows on the sides of the square) by the diagram below, where the vertices are labelled $v_1, \ldots, v_9$.



Consider now the surjective simplicial map $f : K \to L$ determined by

$$f(v_i) = w_{i \bmod 3},$$

where L is a finite simplicial complex with vertices w_0, w_1, w_2.

(i) Sketch the simplicial complex L. State whether L triangulates a closed surface and, if so, identify that closed surface. Provide a brief justification for each part of your answer. **[3]**

(ii) Compute the fundamental groups $\pi_1(K, v_1)$ and $\pi_1(L, w_1)$. **[4]**

(iii) Deduce that the homomorphism $f_* : \pi_1(K, v_1) \to \pi_1(L, w_1)$ induced by f is surjective, but not an isomorphism. **[1]**

SECTION B

5. Let $\mathbb{R}$ be equipped with the standard topology and recall that, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer $\leq x$, so $\lfloor 3.142 \rfloor = 3$. We define

$$f : \mathbb{R} \to \mathbb{R} \text{ by } f(x) = \lfloor x \rfloor,$$

$$g : \mathbb{R} \to \mathbb{R} \text{ by } g(x) = x - \lfloor x \rfloor.$$

(a) Specify the image sets $F = f(\mathbb{R})$ and $G = g(\mathbb{R})$ in $\mathbb{R}$. **[1]**

(b) Write down the definition of continuity for a map between topological spaces, and use it to show that neither f nor g is continuous. **[3]**

(c) We now give each of F and G the subspace topology from $\mathbb{R}$.
Describe these topologies τ_F and τ_G (that is, say which subsets of F and G are open) and briefly explain why. **[3]**

(d) Next we construct the product space $F \times G$, and give it the product topology from τ_F and τ_G. Show that

$$h : \mathbb{R} \to F \times G \text{ given by } h(x) = (f(x), g(x))$$

is a bijection, and specify its inverse $h^{-1} : F \times G \to \mathbb{R}$. **[3]**

(e) Show that h is not continuous. **[2]**

(f) Explain why h^{-1} is continuous. **[3]**

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6. (a) For a topological space X, write down the definitions of i) connectedness, and ii) path-connectedness. **[2]**

(b) Show that if X is path-connected, then it is connected. **[3]**

We now consider the set $M_2(\mathbb{R})$ of 2×2 matrices of real numbers, given the standard topology on $\mathbb{R}^4$ as usual.

(c) Show that $\mathrm{GL}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$ is not connected. **[2]**

Let

$$S = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid \det(A) = 0;\ a, b, c, d \text{ all} \geq 0 \right\} \subseteq M_2(\mathbb{R}) \setminus \mathrm{GL}_2(\mathbb{R})$$

$$\text{and } T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S \mid a, b, c, d \text{ all} > 0 \right\}.$$

(d) Show explicitly, by giving a path, that T is path-connected. **[3]**

(e) A typical matrix in $S \setminus T$ is $L = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ with $c, d > 0$.
Show that L is a limit point for T.
(Hint: Many different metrics induce the standard topology on $\mathbb{R}^4$.) **[3]**

(f) Given that *any* matrix in $S \setminus T$ is a limit point for T, show that S is connected. **[2]**

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7. Let X be the connected, two-dimensional, finite simplicial complex given by $X = K \cup L$, where K and L are connected, two-dimensional, finite simplicial complexes and where the intersection $K \cap L$ is a single 0-simplex common to both K and L. Suppose, in addition, that K triangulates the Klein bottle and that L triangulates the topological space given by removing a small open disc from the real projective plane.

(a) Prove or disprove the statement that X is homeomorphic to a closed surface. Briefly justify any assertions you make. **[2]**

(b) Compute the Euler characteristic of X, justifying any assertions you make. **[5]**

(c) Compute the fundamental group $\pi_1(X)$.
[*If necessary, you may assume knowledge of the fundamental group of the circle S^1 and of any contractible space, but you should present as part of your answer a calculation of the fundamental group of any other space you use.*] **[8]**

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8. (a) Let S_1 and S_2 be two closed surfaces. State the definition of the connected sum $S_1 \# S_2$ of S_1 and S_2. [**2**]

(b) Let $K \# \mathbb{P}$ be the connected sum of the Klein bottle K and the real projective plane $\mathbb{P}$.

(i) Compute the Euler characteristic $\chi(K \# \mathbb{P})$ of $K \# \mathbb{P}$, briefly justifying any formula you use. [**5**]

(ii) Using $K \# \mathbb{P}$ as your starting point, explain how to obtain a closed, orientable surface Σ with Euler characteristic $\chi(\Sigma) = -2$ by attaching or removing discs, handles or crosscaps, as appropriate. You must give a brief explanation of the effect on the Euler characteristic of any operations you perform. [**8**]
