



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH3291-WE01
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Title: Partial Differential Equations III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Consider the PDE

$$\begin{cases} x\partial_x u(x, y) + y\partial_y u(x, y) = 0, & (x, y) \in \mathbb{R}^2, \\ u(x, 5) = x, & x \in \mathbb{R}. \end{cases}$$

- (a) Find the Cauchy curve Γ associated to the problem and all the points on it which are non-characteristics. [3]
- (b) Find a solution to the equation using the method of characteristics and determine its maximal domain of definition. [7]
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2. Consider the PDE

$$\begin{cases} \partial_t u(x, t) + 4u(x, t)^3 \partial_x u(x, t) = 0, & (x, y) \in \mathbb{R}^2, \\ u(x, 0) = \arctan(x), & x \in \mathbb{R}. \end{cases}$$

- (a) Show that the equation is a conservation law by finding a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that our PDE can be written as [3]

$$\partial_t u(x, t) + \partial_x (f(u(x, t))) = 0$$

- (b) Without solving the equation, show that there exists a classical solution to our PDE in $\mathbb{R} \times [0, \infty)$. Justify your answer. [7]
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3. Consider the system

$$u_t(x, t) = u_{xx}(x, t) - u(x, t) \quad \text{and} \quad u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$ and $t \geq 0$.

- (a) Write down a differential equation for the Fourier transform $\hat{u}(\xi, t)$. [2]
- (b) Solve the equation for $\hat{u}(\xi, t)$, and use it to obtain an (explicit) integral representation for $u(x, t)$ in terms of $u_0(x)$. [4]
- (c) Given that $u_0 \in L^2(\mathbb{R})$, show that $u(\cdot, t) \in L^2(\mathbb{R})$ for all $t \geq 0$. Cite any results you use (no need to prove them). [4]
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4. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with smooth boundary, and let $m, f : \Omega \rightarrow \mathbb{R}$ be given with $m(x) > 0$. For $u \in V := \{v \in C^1(\bar{\Omega}) : v = 0 \text{ on } \partial\Omega\}$, we put

$$E[u] := \int_{\Omega} \{|\nabla u(x)|^2 + m(x)u(x)^2 - 2f(x)u(x)\} dx.$$

- (a) Assuming that $u \in V$ is a minimiser of $E[u]$, write down the equation that it must satisfy (showing that u is some weak solution). [4]
- (b) Assuming further that $u \in V \cap C^2(\bar{\Omega})$, write down the partial differential equation and boundary conditions satisfied by u . [4]
- (c) Assuming a minimiser exists, is it unique? Justify your answer. [2]

SECTION B

5. Consider Burgers' equation

$$\begin{cases} \partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0, & (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$u_0(x) = \begin{cases} 0, & x < 0, \\ x + 1, & x > 0. \end{cases}$$

- (a) Find and draw the characteristics associated to the problem. [3]
- (b) Show that

$$u(x, t) = \begin{cases} 0, & x < \sqrt{1+t} - 1, \quad t > 0, \\ \frac{1+x}{1+t}, & x > \sqrt{1+t} - 1, \quad t > 0. \end{cases}$$

is a weak integral solution for our problem. Justify your answer. [10]

- (c) Does the above solution satisfy Lax's entropy condition? Justify your answer. [2]

6. In this question we will see that we can utilise the method of characteristics to solve a linear system of first order differential equations when the leading vector field is the same throughout the system.

Consider the system of equations

$$\begin{cases} 3\partial_x u_1(x, y) + 6x\partial_y u_1(x, y) = u_1(x, y), & (x, y) \in \mathbb{R}^2, \\ 3\partial_x u_2(x, y) + 6x\partial_y u_2(x, y) = u_1^2(x, y), & (x, y) \in \mathbb{R}^2, \\ u_1(0, y) = 1, & y \in \mathbb{R}, \\ u_2(0, y) = y, & y \in \mathbb{R}. \end{cases}$$

- (a) Check that every point on the Cauchy curve $\Gamma = \{(0, y) \mid y \in \mathbb{R}\}$ is non-characteristics with respect to the leading vector field

$$\vec{a}(x, y) = (3, 6x).$$

[2]

(b) Find the characteristics $(X(\tau, s), Y(\tau, s))$ which are associated to the given leading vector field. [4]

(c) Show that

$$z_1(\tau, s) = u_1(X(\tau, s), Y(\tau, s)) \quad \text{and} \quad z_2(\tau, s) = u_2(X(\tau, s), Y(\tau, s)),$$

where $(X(\tau, s), Y(\tau, s))$ are the characteristics found in part (6.b), are given by [5]

$$z_1(\tau, s) = e^\tau, \quad z_2(\tau, s) = s + \frac{e^{2\tau} - 1}{2}.$$

(d) Find a solution to the system of PDEs, $u_1(x, y)$ and $u_2(x, y)$. Where is it defined? [4]

7. Let $\Omega \subset \mathbb{R}^n$ be connected and bounded with smooth boundary.

(a) Prove that if u is harmonic in Ω , then

$$\int_{\partial\Omega} \nabla u \cdot dS = 0.$$

[2]

(b) Use the above fact to prove the mean-value formulas:

$$\frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) dS(y) = u(x) \quad \text{and} \quad \frac{1}{|B_r|} \int_{B_r(x)} u(y) dy = u(x)$$

where $|B_r|$ and $|\partial B_r|$ are the “volume” of a ball and a sphere of radius r . [4]

(c) State and prove the *weak maximum principle* for harmonic functions. [5]

(d) Using the weak maximum principle, prove that a solution u to the problem

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ in } \partial\Omega$$

is unique if it exists. You may assume that $f \in C^2(\bar{\Omega})$ and $g \in C^1(\partial\Omega)$. [4]

8. Consider the problem

$$\begin{aligned} u_t(x, t) &= -u_{xxxx}(x, t) + f(x), \\ u(a, t) &= u(b, t) = u_x(a, t) = u_x(b, t) = 0 \end{aligned}$$

where $x \in [a, b]$ and $t \geq 0$, and $f \in C([a, b])$ and $u(x, 0) = u_0(x)$ are given.

(a) For any $w \in C^2([a, b])$ with $w(a) = w(b) = w'(a) = w'(b) = 0$, prove that

$$\|w\|_{L^2} \leq c_p \|w'\|_{L^2} \leq c_p^2 \|w''\|_{L^2}$$

for some (positive) constant c_p . [3]

- (b) Assuming that u is a classical solution to the above problem, compute a bound for $\|u(\cdot, t)\|_{L^2}$ of the form

$$\|u(\cdot, t)\|_{L^2}^2 \leq e^{-c_1 t} \|u_0\|_{L^2}^2 + c_2 \|f\|_{L^2}^2$$

for some constants $c_1, c_2 > 0$. Hint: multiply by u and integrate by parts. [6]

- (c) Let $v(x)$ be the classical solution of

$$v_{xxxx}(x) = f(x)$$

with $v(a) = v(b) = v_x(a) = v_x(b) = 0$. Show that $u(\cdot, t) \rightarrow v$ in $L^2([a, b])$ as $t \rightarrow \infty$. [6]
