Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3301-WE01

Title:
Mathematical Finance III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider a one-period binomial market with bond process given by $B_0 = 1$ and $B_1 = 1 + r$, and stock process given by $S_0 = 1$ and

$$S_1 = \begin{cases} u & \text{with probability } p_u, \\ d & \text{with probability } p_d, \end{cases}$$

under the real-world probability measure. Here $r > 0$, $u > d > 0$ are constant parameters, and p_u and p_d are positive probabilities that sum to 1.

(a) Show that $d < 1 + r < u$ if and only if the market is arbitrage-free. **[5]**

(b) State the definition of what it means for a measure $\mathbb{Q}$ to be a *martingale measure* on this market. **[2]**

(c) Show that the market is arbitrage-free if and only if there exists a unique martingale measure $\mathbb{Q}$. **[3]**

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2. Consider a one-period binomial market with bond process given by $B_0 = 1$ and $B_1 = 1.1$, and stock process given by $S_0 = 1$ and

$$S_1 = \begin{cases} 1.2 & \text{with probability } 1/5, \\ 0.8 & \text{with probability } 4/5, \end{cases}$$

under the real-world probability measure.

(a) Compute the replicating portfolio consisting of bonds and stocks for a call option with exercise time 1 and strike price $K_1 \in (0.8, 1.2)$. Hence find the time 0 fair price of the call option. **[5]**

(b) Using the risk-neutral evaluation formula, or otherwise, find the fair price of a put option with exercise time 1 and strike price $K_2 \in (0.8, 1.2)$. **[5]**

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3. In this question, $(W_t, t \geq 0)$ denotes a standard $\mathbb{R}$-valued Brownian motion.

(a) State the defining properties of Brownian motion. **[3]**

(b) Explain why a Brownian motion is a *martingale* with respect to its natural filtration. **[3]**

(c) Let $a > 0$ be a constant, and set

$$Y_t := \frac{1}{2}\big(W_{4t+a} - W_a\big), \quad t \geq 0.$$

Prove that $(Y_t, t \geq 0)$ is a Brownian motion. **[4]**

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4. In this question, $(W_t, t \geq 0)$ denotes a standard $\mathbb{R}$-valued Brownian motion. Define stochastic processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ by the Itô integrals

$$X_t := \int_0^t s \, \mathrm{d}W_s, \quad Y_t := \int_0^t (t-s) \, \mathrm{d}W_s, \quad t \geq 0.$$

(a) Show that $X_t + Y_t = tW_t$ for all $t > 0$. [**1**]

(b) What is the distribution, for fixed $t \geq 0$, of X_t? [**4**]

(c) What is the distribution, for fixed $t \geq 0$, of Y_t? [**3**]

(d) Compute $\mathbb{C}\text{ov}(X_t, Y_t)$ for $t \geq 0$. [**2**]

SECTION B

5. Consider a two-period recombinant binomial model with initial stock price $S_0 = 8$, risk-free interest rate $1/4$, and constant factors $u = 2$ and $d = 1/2$ for the price dynamics.

(a) Calculate the martingale probabilities at each branch in the tree. **[2]**

A *European lookback put option* with exercise time $T = 2$ has payoff

$$\Phi^{\mathrm{E}}_{\mathrm{LP}}(S_0, S_1, S_2) = S_2^{\star} - S_2,$$

where $S_t^{\star} := \max(S_u : 0 \le u \le t)$ is the maximum price up to time t.

(b) Find the no-arbitrage price $\Pi^{\mathrm{E}}_{\mathrm{LP}}$ at time 0 for the European lookback put option with payoff function $\Phi^{\mathrm{E}}_{\mathrm{LP}}$. **[5]**

An *American* lookback put option has payoff

$$\Phi^{\mathrm{A}}_{\mathrm{LP}}(S_0, S_1, S_2) = S_\tau^{\star} - S_\tau,$$

where τ is a stopping time corresponding to a strategy freely determined (among all stopping times) by the holder of the option.

(c) Compute the no-arbitrage price $\Pi^{\mathrm{A}}_{\mathrm{LP}}$ of this option. **[8]**

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6. Consider an arbitrage-free multi-period binomial market with bond process $B_t = (1 + r)^t$ and stock process S_t, $t \in \{0, 1, \ldots, T\}$.

Let C_0^{E} and P_0^{E} denote the no-arbitrage prices at time 0 of European call and, respectively, put options on this stock, with the same exercise time T and strike price K.

(a) By using the absence of arbitrage, or otherwise, prove that

$$C_0^{\mathrm{E}} - P_0^{\mathrm{E}} = S_0 - K(1 + r)^{-T},$$

which is a version of the *put-call parity* formula. **[7]**

In addition to the European options already defined, let C_0^{A} and P_0^{A} denote the no-arbitrage prices at time 0 of American call and, respectively, put options, also with the same exercise time T and strike price K.

(b) Suppose you hold a portfolio with one American call option and short one American put option, each with strike price K.
For time $t \le T$, what is the time t payoff of the portfolio if both options are exercised at time t? **[2]**

(c) By using the absence of arbitrage, or otherwise, prove $C_0^{\mathrm{A}} - P_0^{\mathrm{A}} \ge S_0 - K$. **[6]**

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7. Consider a Black–Scholes market with risk-free asset given by $B_t = \mathrm{e}^{rt}$, $r > 0$, and two risky assets with evolution given by $S_0^{(1)} = S_0^{(2)} = 1$ and

$$\mathrm{d}S_t^{(1)} = \mu_1 S_t^{(1)}\,\mathrm{d}t + \sigma_1 S_t^{(1)}\,\mathrm{d}W_t^{(1)}, \quad \text{and} \quad \mathrm{d}S_t^{(2)} = \mu_2 S_t^{(2)}\,\mathrm{d}t + \sigma_2 S_t^{(2)}\,\mathrm{d}W_t^{(2)}.$$

Here $\mu_1, \mu_2, \sigma_1, \sigma_2$ are positive constants, and

$$W_t^{(1)} = W_t, \quad \text{and} \quad W_t^{(2)} = \rho W_t + \sqrt{1-\rho^2}\,W_t',$$

where $\rho \in (-1,1)$ is constant and $(W_t, t \geq 0)$, $(W_t', t \geq 0)$ are independent Brownian motions under the real-world measure $\mathbb{P}$.

Set $R_t := S_t^{(1)}/S_t^{(2)}$ for $t \geq 0$.

(a) Using Itô's formula, derive SDEs for $L_t^{(1)} := \log S_t^{(1)}$ and $L_t^{(2)} := \log S_t^{(2)}$. **[4]**

(b) Using your answer to question (**7.**a), or otherwise, write down an SDE for $\log R_t$. Deduce that

$$\mathrm{d}R_t = \mu R_t\,\mathrm{d}t + \sigma R_t\,\mathrm{d}W_t'',$$

where $(W_t'', t \geq 0)$ is a Brownian motion, and μ and σ are constant functions of μ_1, μ_2, σ_1, σ_2 and ρ which you should determine. **[6]**

(c) If $\mathbb{Q}$ is a measure under which $(\mathrm{e}^{-rt}R_t, t \geq 0)$ is a martingale, find an expression for $\mathbb{Q}(R_T \geq \theta)$, where $\theta > 0$, $T > 0$, in terms of the function $\overline{N}(z) := \mathbb{P}(W_1 \geq z)$. **[5]**

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8. Consider the continuous-time Black–Scholes market, with price dynamics given by

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \qquad \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where $r > 0$ is the risk-free interest rate, μ and σ are constant parameters, and $(W_t, t \geq 0)$ is a Brownian motion under the real-world measure $\mathbb{P}$.

Define contingent claims

$$X_T = S_T\,\mathbb{1}\{S_{T/2} \geq K\}, \quad \text{and} \quad Y_T = S_T\,\mathbb{1}\{S_T \geq K\},$$

for constants $T, K > 0$, where $\mathbb{1}\{\,\cdot\,\}$ denotes the indicator random variable that takes the value 1 if the indicated event occurs, and 0 if not.

(a) Writing the expectation using the probability density function of W_1, or otherwise, obtain a formula for

$$\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{bW_1}\,\mathbb{1}\{W_1 \geq a\}\right]$$

in terms of a, b and the function $\overline{N}(z) := \mathbb{P}(W_1 \geq z)$, $z \in \mathbb{R}$. **[4]**

(b) Show that, if $\mathbb{Q}$ is the risk-neutral measure for this market, then

$$\mathbb{E}_{\mathbb{Q}}(Y_T) = \mathrm{e}^{rT} S_0 \overline{N}\left(\frac{\log(K/S_0) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

and hence give a formula for $\Pi_0(Y_T)$, the arbitrage-free price of Y_T at time 0, in terms of $\overline{N}$, K, T, and appropriate market parameters. **[5]**

(c) Calculate a formula for $\Pi_0(X_T)$, the arbitrage-free price of X_T at time 0, in terms of $\overline{N}$, K, T, and appropriate market parameters. *Hint:* You may wish to relate X_T to $Y_{T/2}$ and make use of the expression in part (**8.**b). **[6]**
