Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH33020-WE01

Title: Geometric Topology V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

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SECTION A

1. (a) Let X be a topological space. Give the definition of a path-homotopy between two paths $\alpha, \beta\colon [0,1] \to X$ in a topological space. **[3]**

(b) Suppose $\gamma\colon [0,1] \to X$ is a path in a topological space X. Give the definition of the inverse path γ^- of the path γ. **[1]**

(c) Given two paths $\gamma, \mu\colon [0,1] \to X$ in a topological space X such that $\gamma(1) = \mu(0)$. Give the definition of the composite path $\gamma * \mu$. **[1]**

(d) Suppose γ is a path with $\gamma(0) = x_0$. Let $\underline{x_0}\colon [0,1] \to X$ be the path which is constant to x_0 for all $t \in [0,1]$. Show that $\gamma * \gamma^-$ is path-homotopic to $\underline{x_0}$ by specifying a homotopy. You do not need to prove continuity of your suggested homotopy, but the expression must be correct. **[5]**

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2. In this problem the sphere S^n denotes the set of points

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \|(x_1, \ldots, x_{n+1})\| = 1\}.$$

(a) For $n, m \geq 1$, give the definition of an antipodal map $f\colon S^n \to S^m$. **[2]**

(b) Suppose $n = 2$. For each of the values $m = 1, 2, 3$, determine whether or not there exists an antipodal map $f\colon S^n \to S^m$, and for which it does not. Justify your answer. If you conclude that there exists one, then provide an example of such an antipodal map. **[8]**

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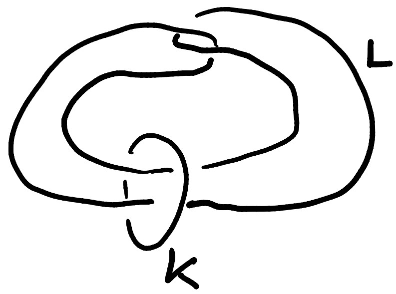
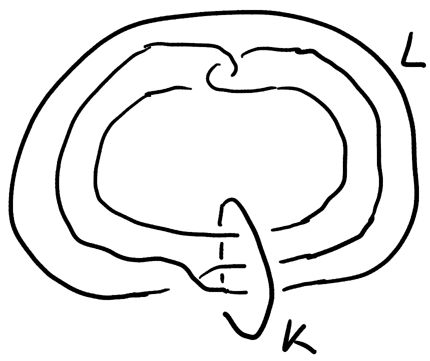
3. Determine whether each of the following statements is true or false. Provide a sketch of proof of your claim. You are allowed to use properties of examples that appeared in the class or in the homework problems.

(a) The fundamental group of the complement of the unknot $U \subseteq \mathbb{R}^3$ is the free abelian group isomorphic to $\mathbb{Z}$. **[3]**

(b) The component L of the following link represents the trivial element in $\pi_1(S^3 \setminus K, x_0)$, no matter how L is oriented to represent an oriented loop at a point $x_0 \in L$. **[3]**



(c) The component L of the following link represents the trivial element in $\pi_1(S^3 \setminus K, x_0)$, no matter how L is oriented to represent an oriented loop at a point $x_0 \in L$. **[4]**



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4. In this problem the group A_n denotes the alternating group on n elements, the subgroup of the permutation group S_n of even permutations.

(a) State the definition of a torus knot. **[2]**

(b) Draw the torus knot $T_{2,5}$. **[2]**

(c) Show that the torus knot $K = T_{3,5}$ admits a homomorphism

$$\rho\colon \pi_1(S^3 \setminus K, x_0) \to A_5$$

with non-abelian image. Here x_0 is an arbitrary base point in the complement of K. Justify your answer by stating a presentation of $\pi_1(S^3 \setminus K, x_0)$, and by showing that there are elements in the image of ρ which do not commute. **[6]**

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SECTION B

5. In this problem, let n be a positive integer.

(a) Give the definition of the n-dimensional real projective space $\mathbb{RP}^n$. **[2]**

(b) For every n, choose a point $x_0 \in \mathbb{RP}^n$, and provide a generator of the fundamental group $\pi_1(\mathbb{RP}^n, x_0)$. Depending on the integer n, state to which of the abelian groups

$$\mathbb{Z}, \text{ or } \mathbb{Z}/k \text{ for some } k \in \mathbb{N}$$

the fundamental group $\pi_1(\mathbb{RP}^n, x_0)$ is isomorphic to. You do not need to justify your answer. **[3]**

(c) State the definition of a retract $A \subseteq X$ of a topological space X. **[1]**

(d) Decide for which $n \geq 1$ there is a retract $A \subseteq \mathbb{RP}^n$ which is homeomorphic to the circle S^1, that is $A \cong S^1$. In the affirmative case, provide such a retract. Justify your answer. **[4]**

(e) Give a definition of the Klein bottle. **[1]**

(f) Determine a retract of the Klein bottle that is homeomorphic to S^1. Justify your answer. **[4]**

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6. Let $K \subseteq \mathbb{R}^3$ be a knot. Let $\mathbb{R}^3 \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^3$. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\}$ be the natural inclusion map. Let x_0 be a point in $\mathbb{R}^3 \setminus K$.

(a) State what topology the one-point compactification of $\mathbb{R}^3$ admits by providing all its open sets. **[2]**

(b) State the Seifert-van-Kampen theorem. **[4]**

(c) State the fundamental groups of the spheres S^n for any $n \geq 1$. You do not need to justify your answer. **[1]**

(d) Using the Seifert-van-Kampen theorem, prove that the inclusion map

$$h\colon \mathbb{R}^3 \setminus K \to \mathbb{R}^3 \cup \{\infty\} \setminus f(K)$$

yields an isomorphism of fundamental groups, that is, the map

$$h_*\colon \pi_1(\mathbb{R}^3 \setminus K, x_0) \to \pi_1(\mathbb{R}^3 \cup \{\infty\} \setminus f(K), f(x_0))$$

is an isomorphism. **[8]**

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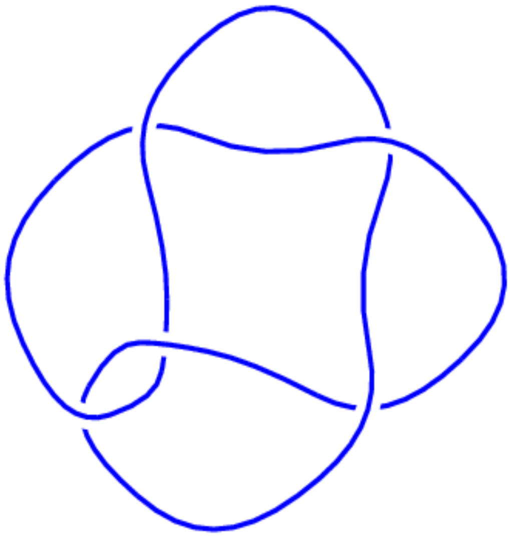
7. In this exercise we study the knot K given by the following diagram.



(a) State the definition of a Seifert surface of a knot. **[2]**

(b) Draw a Seifert surface of the knot K above. **[3]**

(c) Compute the Alexander polynomial of the knot K. **[7]**

(d) State the definition of the genus of a knot. **[1]**

(e) Compute the genus of the knot K. **[2]**

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8. (a) Provide a knot of genus 2, and justify your answer using the Alexander polynomial. **[8]**

(b) Provide a diagram of a non-trivial knot where Seifert's algorithm yields a Seifert surface of non-minimal genus. Justify your answer. **[7]**

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