EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH3391-WE01

Title: Quantum Computing III

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider the two-qubit Hilbert space with computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In this basis define the operator

$$\hat{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

(a) Show that $\hat{M}$ is a valid observable for this system. Compute its eigenvalues and eigenvectors. **[4]**

(b) Let the system be prepared in the state

$$|\chi\rangle = \frac{1}{\sqrt{6}}\big(2\,|00\rangle + |10\rangle + |11\rangle\big).$$

Compute the probability of each distinct measurement outcome of $\hat{M}$. **[4]**

(c) Suppose a measurement of $\hat{M}$ is performed on the state $|\chi\rangle$ and the outcome corresponds to the largest eigenvalue of $\hat{M}$. Determine the post-measurement state. **[2]**

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2. A one-qubit system is described by the density matrix

$$\hat{\rho} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

(a) Verify that $\hat{\rho}$ is a valid density matrix. Then compute its Bloch vector. **[3]**

(b) Determine whether this state is pure or mixed. Verify that the Bloch vector obtained above is consistent with this answer. **[3]**

(c) Find the eigenvalues of $\hat{\rho}$. Using these, compute the Von Neumann entropy. Argue that the result is compatible with your answer in (b). **[4]**

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3. (a) Given that NOT, AND, CNOT form a universal gate set for classical computation, show that quantum computation contains classical computation; that is, that any classical computation can be implemented as a unitary operator acting on a Hilbert space $\mathcal{H}$. **[5]**

(b) Construct a quantum circuit realising the function

$$f(x_2x_1x_0) = x_2 + x_1 + x_0 \mod 2\,,$$

where $x_2x_1x_0$ is a three-bit number. **[5]**

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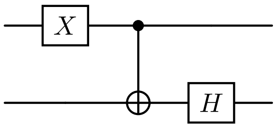
4. Consider the quantum circuit



(a) Determine its action on the computational basis states. [**6**]

(b) Write the action as a 4×4 unitary matrix acting on the computational basis states. [**2**]

(c) Construct a circuit that implements the inverse transformation. [**2**]

SECTION B

5. Consider a 3-qubit system ABC in the mixed state described by the density matrix

$$\hat{\rho}_{ABC} = \frac{1}{2} |\psi\rangle \langle\psi| + \frac{1}{2} |000\rangle \langle 000| \ ,$$

where the pure state $|\psi\rangle$ is defined as:

$$|\psi\rangle = \frac{1}{2} |000\rangle + \frac{\sqrt{3}}{2} |111\rangle \ .$$

(a) Compute the reduced density matrix $\hat{\rho}_{AB} = \mathrm{Tr}_C(\hat{\rho})$. Determine its eigenvalues and express the Von Neumann entropy $S(AB)$ in terms of $H_2(p)$, where

$$H_2(p) := -p \log_2 p - (1-p) \log_2(1-p) \qquad (5.1)$$

is the *binary entropy function*. **[4]**

(b) Also compute the reduced density matrix $\hat{\rho}_A = \mathrm{Tr}_{BC}(\hat{\rho})$ and the associated Von Neumann entropy $S(A)$ in terms of $H_2(p)$. **[3]**

(c) Finally calculate the non-zero eigenvalues of the density matrix $\hat{\rho}_{ABC}$ and the global Von Neumann entropy $S(ABC)$, again in terms of $H_2(p)$. **[5]**

(d) Compare $S(ABC)$ and $S(AB)$. Use the properties of the binary entropy function $H_2(p)$ to determine which is larger. Use this result together with the symmetry of the system to show that the system satisfies

$$S(AB) + S(BC) \geq S(ABC) + S(B) \, .$$

(the so-called *strong sub-additivity* condition). **[3]**

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6. Alice and Bob share a maximally entangled Bell state $|\beta_{00}\rangle$. Alice intends to teleport an unknown, normalised, pure state $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob. Before Alice performs any operations, the total state of the three-qubit system is

$$|\Psi\rangle = |\psi\rangle \otimes |\beta_{00}\rangle \ .$$

In what follows you will need the definition of the Bell states,

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\bar{y}\rangle \Big) . \qquad (6.1)$$

(a) Write down the reduced density matrix $\hat{\rho}_B$ for Bob's qubit and calculate the Von Neumann entropy $S(\hat{\rho}_B)$. What does this value imply about the information Bob has regarding Alice's state $|\psi\rangle$? **[5]**

(b) Use the definition of the Bell states (6.1) to write the state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{x,y\in\{0,1\}} |\beta_{xy}\rangle \otimes |\psi_{xy}\rangle \ ,$$

for some $|\psi_{xy}\rangle$. Determine these $|\psi_{xy}\rangle$. Alice now performs a measurement which collapses her 2-qubit state to one of the Bell states. Construct an operator corresponding to the observable which she measured. **[4]**

(c) Alice needs to transmit the result of her observation to Bob, but a classical communication failure occurs, and Bob does not receive the measurement outcome. Show that Bob's updated reduced density matrix $\hat{\rho}'_B$ remains identical to the one found in part (a). Explain why this result is necessary to avoid faster-than-light communication. **[3]**

(d) Suppose the communication channel is restored. Alice sends her result, and Bob finds that Alice measured the state $|\beta_{11}\rangle$. Identify the unitary operator $\hat{U}$ that Bob must apply to his qubit to recover $|\psi\rangle$. Calculate the Von Neumann entropy of Bob's qubit after he successfully applies the correction and recovers $|\psi\rangle$. **[3]**

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7. Given a unitary transformation acting on a two-qubit Hilbert space in the computational basis

$$U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

(a) Write U as a product of unitaries U_i which each act non-trivially only on a two-dimensional subspace of the Hilbert space. **[6]**

(b) Decompose each of the U_i in terms of controlled-unitary transformations and single-qubit unitaries. **[6]**

(c) Construct a quantum circuit which realises U. **[3]**

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8. We are given a function $f : \{0,1\}^n \to \{0,1\}$ that is either constant ($f(x)$ has the same value, either or 0 or 1, for all inputs), or *balanced*, meaning that $f(x) = 0$ for half the possible input values x and $f(x) = 1$ for the other half. In the balanced case we don't know anything about which values x produce which output. We want to determine whether the function is constant or balanced.

(a) Show that classically this problem is not in P but is in BPP. **[3]**

(b) We want to construct a quantum algorithm for this problem. Suppose we have a unitary U_f such that $U_f |x\rangle |m\rangle = |x\rangle |m \oplus f(x)\rangle$, where $|x\rangle$ is an n-qubit state in the computational basis, and $|m\rangle$ is a state of a single ancillary qubit. Show that

$$|\psi\rangle = U_f H^{\otimes(n+1)} |0\rangle_n |1\rangle$$

has the form $|\psi\rangle = |\phi\rangle |\chi\rangle$, where

$$|\phi\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} (-1)^{f(y)} |y\rangle$$

and $|\chi\rangle$ is a single-qubit state which you should determine. **[7]**

(c) Now consider acting on the n-qubit Hilbert space with $H^{\otimes n}$ again. Show that if we perform a measurement on the resulting state $|\rho\rangle = H^{\otimes n} |\phi\rangle$ in the computational basis, the probability that we get the zero state is $P(0) = 1$ if $f(x)$ is constant, and $P(0) = 0$ if $f(x)$ is balanced. **[5]**
