



## EXAMINATION PAPER

<b>Examination Session:</b> May/June	<b>Year:</b> 2026	<b>Exam Code:</b> MATH3471-WE01
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<b>Title:</b> Geometry of Mathematical Physics III
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Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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<b>Revision:</b>	
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## SECTION A

1. Consider the group  $G = U(2)$  with respect to matrix multiplication. In each of the following cases, explain whether the given map defines a representation of  $G$ , and if so, whether the representation is faithful and irreducible.

(a) Let  $V = \mathbb{C}$  and define

$$\begin{aligned} r_1 : G &\rightarrow GL(V), \\ r_1(g) &= \det(g). \end{aligned} \quad [5]$$

(b) Let  $V = \mathbb{C}^4$  and define

$$\begin{aligned} r_2 : G &\rightarrow GL(V), \\ r_2(g) &= \begin{pmatrix} g & \mathbb{1}_2 \\ \mathbf{0} & g \end{pmatrix}. \end{aligned}$$

Here  $\mathbb{1}_2$  denotes the  $2 \times 2$  identity matrix while  $\mathbf{0}$  denotes the  $2 \times 2$  zero matrix. [5]

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2. Let us define

$$H = \left\{ \begin{pmatrix} g_1 & \mathbf{0} \\ \mathbf{0} & g_2 \end{pmatrix} \mid g_1, g_2 \in SU(2) \right\},$$

where  $\mathbf{0}$  denotes the  $2 \times 2$  zero matrix.

- (a) Show that  $H$  is a subgroup of  $SU(4)$ . [5]  
(b) Consider the fundamental representation of  $SU(4)$

$$R : SU(4) \rightarrow GL(4, \mathbb{C}), \quad R(U) = U, \quad \forall U \in SU(4).$$

Show that by restricting  $R$  to  $H$  we obtain a representation of  $H$ . Determine whether or not this representation of  $H$  is irreducible. [5]

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3. The field strength tensor of electromagnetism is given in terms of electric and magnetic fields by

$$F^{\mu\nu} := \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

Write the following expressions in terms of electric and magnetic fields:

- (a)  $F_{\mu\nu}$  [2]  
 (b)  $F^{\mu\nu} F_{\mu\nu}$  [4]  
 (c)  $F^{\mu\nu} F^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma}$  [4]
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4. Let  $\Phi$  transform under the gauge group  $U(1)$  as

$$\Phi \rightarrow e^{in\theta(\mathbf{x})} \Phi$$

for  $n \in \mathbb{Z}$ .

- (a) Work out the transformation of  $\partial_\mu \Phi$ . [3]  
 (b) Write down the covariant derivative  $D_\mu \Phi$  and show that it has the correct transformation behaviour. [3]  
 (c) Consider the following action for  $\Phi$ :

$$S = - \int d^4x \left[ D_\mu \Phi \overline{D^\mu \Phi} + U(|\Phi|^2) + \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right].$$

Find the conserved current using Noether's theorem. [4]

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## SECTION B

5. Let us define

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

- (a) Show that  $G$  forms a group with respect to matrix multiplication which you may assume is associative. [5]
- (b) Find a basis for the Lie algebra  $\mathfrak{g}$  of the group  $G$  and compute its structure constants. [5]
- (c) Show that the exponential map

$$\exp : \mathfrak{g} \rightarrow G$$

is surjective, that is, show that every element of  $G$  can be written as  $e^\gamma$  for some  $\gamma \in \mathfrak{g}$ . [5]

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6. Let  $\mathfrak{g} = \text{span}_{\mathbb{R}}\{\mathbf{x}, \mathbf{y}\}$  be an abstract two-dimensional real Lie algebra spanned by the basis elements  $\mathbf{x}, \mathbf{y}$ .

- (a) Show that if  $[\mathbf{x}, \mathbf{y}] = 0$  then  $\mathfrak{g}$  must be Abelian, i.e.

$$[\mathbf{v}, \mathbf{w}] = 0, \quad \forall \mathbf{v}, \mathbf{w} \in \mathfrak{g}. \quad [4]$$

- (b) Let us assume now that  $[\mathbf{x}, \mathbf{y}] \neq \mathbf{0}$ . Construct a basis  $\{\mathbf{a}, \mathbf{b}\}$  for  $\mathfrak{g}$  such that

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a}.$$

[Hint: use  $[\mathbf{x}, \mathbf{y}]$  itself to help construct the required basis for  $\mathfrak{g}$ .] [6]

- (c) Let  $\mathfrak{g} = \text{span}_{\mathbb{R}}\{\mathbf{a}, \mathbf{b}\}$  be the same Lie algebra and basis as in part (b). Consider the linear map  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(2, \mathbb{R})$  defined on a general Lie algebra element  $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} \in \mathfrak{g}$  as

$$\rho(\mathbf{v}) = \rho(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\rho(\mathbf{a}) + \beta\rho(\mathbf{b}), \quad \alpha, \beta \in \mathbb{R},$$

where

$$\rho(\mathbf{a}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(\mathbf{b}) = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that  $\rho$  defines a Lie algebra representation for the Lie algebra  $\mathfrak{g}$ . [5]

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7. Let  $\phi$  be a complex scalar field with Lagrangian

$$\mathcal{L} = \overline{\partial_\mu \phi} \partial^\mu \phi.$$

- (a) State the transformation of a scalar field under Lorentz transformations and explain why the action above defines a Lorentz invariant field theory. [4]
- (b) Find the equations of motion and show that they are solved by

$$\phi = e^{ik_\mu x^\mu},$$

whenever  $k_\mu k^\mu = 0$ . [4]

- (c) How does the equation of motion change if we include a term  $m^2|\phi|^2$  in the Lagrangian? Under which condition is  $\phi = e^{ik_\mu x^\mu}$  a solution now? [2]
- (d) A general solution to the equations of motion for  $m = 0$  is given by

$$\phi(\mathbf{x}) = \int d^4k \delta(k^\mu k_\mu) e^{ik_\mu x^\mu} f(\mathbf{k}),$$

for an arbitrary smooth function  $f(\mathbf{k})$ . Use this to show that the field theory in question is Lorentz invariant. [5]

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8. Let us consider the following Lagrangian for a real scalar  $\phi$  and three Dirac spinors  $\Psi_i$ :

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_{i=1}^3 (\bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i + y_i \bar{\Psi}_i \Psi_i \phi) .$$

- (a) Find the equations of motion. [5]

- (b) Using the defining representation, we can act with the group  $U(3)$  on the  $\Psi_i$ , i.e. for  $U \in U(3)$  we let  $\Psi_i \rightarrow U_{ij} \Psi_j$ . In each of the following cases, decide which  $U \in U(3)$  constitute global symmetries of  $\mathcal{L}$ :

- (i)  $y_1 = y_2 = y_3$ ,  
(ii)  $y_1 = y_2 \neq y_3$ ,  
(iii)  $y_i \neq y_j \forall i \neq j$ . [4]

- (c) Assume that the two further terms

$$\delta \mathcal{L} = \text{tr} (\partial_\mu \Omega \partial^\mu \Omega) + \sum_{i,j=1}^3 \bar{\Psi}_i \Omega_{ij} \Psi_j$$

are included in the action, and that the field  $\Omega$  is a matrix (with components  $\Omega_{ij}$ ) which transforms in the adjoint representation of  $U(3)$ . How does this change the analysis you have done in part (b) ? [3]

- (d) Let  $\Omega = \sum_a \Omega_a t_a$  with  $t_a$  a basis of  $\mathfrak{u}(3)$  such that  $\text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$ . Determine the range of  $a$  and write  $\delta \mathcal{L}$  in terms of the components  $\Omega_a$ . [3]