Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4051-WE01

Title:
General Relativity IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. A two-dimensional spacetime has coordinates (x, y). New local coordinates are defined by $p = ye^x$, $q = e^x$.

(a) A vector field V^μ has components $V^\mu = (\exp(-x), 0)$ with respect to the original coordinates. Calculate its components $\tilde{V}^\mu$ with respect to the new coordinates. **[5]**

(b) A $(1, 1)$ tensor field $T^\mu_{\ \nu}$ has $T^y_{\ y} = 1$, all other components vanishing with respect to the original coordinates. Calculate its components $\tilde{T}^\mu_{\ \nu}$ with respect to the new coordinates. **[5]**

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2. In a spacetime with coordinates (t, x, y, z) and metric $ds^2 = -x^2dt^2 + dx^2 + (1 + 2z^2)dy^2 + dz^2$, a curve is defined by $x^\mu = (\lambda, \sinh\lambda, \lambda, \lambda^2)$ for $\lambda \in (0, 1)$.

(a) Compute the tangent vector V^μ to the curve, and calculate its norm. **[4]**

(b) Say whether the curve is timelike or spacelike, and calculate the proper time or proper length along the curve. **[4]**

(c) Calculate $V^\mu n_\mu$, where n_μ is the normal to the surface $t = y$. **[2]**

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3. Consider the following spacetime metric

$$ds^2 = -\left(1 - \frac{8}{r} + \frac{12}{r^2}\right) dt^2 + \left(1 - \frac{8}{r} + \frac{12}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) .$$

An observer at a fixed position $r = 1$ sends a lightray radially to an observer fixed at $r = 1/2$. How is the frequency measured by the observer at $r = 1/2$ related to the frequency emitted by the observer at $r = 1$? Is this a red-shift or a blue-shift? **[10]**

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4. Consider the following spacetime metric

$$ds^2 = -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) dt^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where M and Q are arbitrary positive constants.

(a) List all coordinate and curvature singularities. Justify your answer, but you do not need to decide if it is a curvature or coordinate singularity. Are all singularities always there? **[5]**

(b) Is $r = M/2$, with $\theta = \pi/2$ and $\phi = 0$, a possible path for a massive observer? Justify your answer. **[5]**

SECTION B

5. Consider a spacetime with metric

$$ds^2 = \exp(2y)(-dt^2 + dx^2) + dy^2.$$

(a) Compute the Christoffel symbols for this spacetime. **[5]**

(b) Identify two Killing vectors for this spacetime, and find the associated conserved quantities for a geodesic $x^\mu(\lambda)$. **[3]**

(c) Find the form of $x^\mu(\lambda)$ for null geodesics in terms of the conserved quantities. Show that null geodesics can reach $y \to -\infty$ in finite affine parameter λ. Determine the behaviour of t, x in this limit. **[7]**

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6. A conformal Killing vector field is a vector field K^μ that satisfies the equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = f(x^\lambda) g_{\mu\nu},$$

where $f(x^\lambda)$ is some function on spacetime.

(a) Suppose that K^μ is a conformal Killing vector field and V^μ is the tangent vector to a geodesic $x^\mu(\lambda)$. Calculate the derivative of $Q = V^\mu K_\mu$ along the geodesic. Under what circumstances is Q constant along the geodesic? **[5]**

(b) Show that if K^μ is a conformal Killing vector, then

$$\nabla_\mu \nabla_\sigma K^\mu = A R_{\lambda\sigma} K^\lambda + B \nabla_\sigma f,$$

where A and B are constants you should determine. **[4]**

(c) Show that for any tensor $T^{\mu\nu}$, $[\nabla_\mu, \nabla_\nu] T^{\mu\nu} = 0$. (You may use without proof that $[\nabla_\mu, \nabla_\nu] T^{\rho\sigma} = R^\rho{}_{\lambda\mu\nu} T^{\lambda\sigma} + R^\sigma{}_{\lambda\mu\nu} T^{\rho\lambda}$.) **[2]**

(d) Hence show that if K^μ is a conformal Killing vector, then

$$\nabla_\mu \nabla_\nu \nabla^\mu K^\nu = C \nabla_\mu \nabla^\mu f,$$

where C is a constant you should determine. **[4]**

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7. The Friedmann Robertson Walker metric is given by

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right) ,$$

where $k = 0, \pm 1$ and $a(t) \geqslant \epsilon$, where ϵ is a small but non-zero positive number. Consider the Friedmann equations for pressure-less matter ($p = 0$):

$$\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{k}{2a} = 0 , \qquad \rho = -\frac{3\ddot{a}}{4\pi a} ,$$

and where we want the density ρ to be positive.

(a) Show that $\ddot{a}a^2 = -c$, where c is a positive constant. **[6]**

(b) Using the Friedmann equations describe the behavior of $a(t)$ for the cases $k = 1$ and $k = 0$. You can use the result of part (a). Does $a(t)$ achieve a maximum value for $k = 0$ or $k = 1$? If so compute the maximum value and determine how much coordinate time it takes to achieve it. You can leave the coordinate time as a definite integral. **[9]**

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8. Consider the following action in two spacetime dimensions:

$$S[g_{\mu\nu}, \Phi] = \int d^2x \sqrt{-g}\, R + \int d^2x \sqrt{-g}\, \Phi(x)(R - 2) ,$$

where $\Phi(x)$ is a scalar field. Obtain the equations of motion for Φ and $g_{\mu\nu}$. You may use the following variations without proof:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} ,$$

and

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\sigma \left(g_{\mu\nu} \nabla^\sigma (\delta g^{\mu\nu}) - \nabla_\lambda (\delta g^{\sigma\lambda}) \right) .$$

When integrating by parts you can ignore boundary terms. **[15]**
