Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider the interacting theory of a complex scalar field $\varPhi(x)$ with the action

$$S[\varPhi, \varPhi^*] = \int d^4x \Big(-\partial_\mu \varPhi^* \partial^\mu \varPhi - m^2 \varPhi^* \varPhi - \frac{\lambda}{4} \varPhi^{*2} \varPhi^2 \Big) .$$

(a) Show that the action is invariant under the transformation

$$\varPhi \to e^{i\alpha} \varPhi , \qquad \varPhi^* \to e^{-i\alpha} \varPhi^* , \tag{1.1}$$

where α is a constant parameter. **[2]**

(b) Compute the conserved Noether current associated with this symmetry. **[4]**

(c) Consider a second complex scalar field $\varPsi$ transforming as

$$\varPsi \to e^{-2i\alpha} \varPsi , \qquad \varPsi^* \to e^{2i\alpha} \varPsi^* . \tag{1.2}$$

Let n_1, n_2, n_3 and n_4 be non-negative integers. Determine two linearly independent cubic interaction terms of the form

$$\sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1+n_2+n_3+n_4=3}} \alpha_{n_1,n_2,n_3,n_4} \varPhi^{n_1} \varPhi^{*n_2} \varPsi^{n_3} \varPsi^{*n_4} ,$$

that are both real and invariant under the transformations (1.1) and (1.2). **[4]**

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2. Consider the free theory of a real scalar field $\phi(x)$ with the action

$$S[\phi] = \int d^4x \Big(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \Big) .$$

Canonically quantizing, the normal ordered Hamiltonian, spatial momentum and number operators can be written in terms of creation and annihilation operators as

$$H = \int \frac{d^3k}{(2\pi)^3} \, \omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} , \qquad \vec{P} = \int \frac{d^3k}{(2\pi)^3} \, \vec{k} a^\dagger_{\vec{k}} a_{\vec{k}} , \qquad \mathcal{N} = \int \frac{d^3k}{(2\pi)^3} \, a^\dagger_{\vec{k}} a_{\vec{k}} ,$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$ and $\vec{k} = (k_1, k_2, k_3)$.

(a) Write down the canonical commutation relations for the creation and annihilation operators. **[1]**

(b) Define the vacuum state $|0\rangle$ of the theory and briefly explain how to construct single particle and multiparticle states. **[3]**

(c) Show that

$$\int \frac{d^3k}{(2\pi)^3} \, f(\vec{k}) [a^\dagger_{\vec{k}} a_{\vec{k}}, a^\dagger_{\vec{p}}] = f(\vec{p}) a^\dagger_{\vec{p}} ,$$

where $f(\vec{k})$ is an arbitrary function of $\vec{k}$. **[2]**

(d) Consider the state

$$(a^\dagger_{\vec{p}_1} + a^\dagger_{\vec{p}_2}) |0\rangle .$$

Determine for which $\vec{p}_1$ and $\vec{p}_2$ this state is an eigenstate of the normal ordered Hamiltonian, spatial momentum and number operators. You should find a different result for each operator. **[4]**

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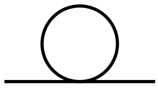
3. Consider ϕ^4 theory in $d = 4$ spacetime dimensions:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4.$$

(a) The 1-loop correction to the self-energy is given by the following off-shell amputated diagram:

$\mathcal{A}_2 =$ 

Show that this is given by

$$\mathcal{A}_2 = -\frac{g}{2}\int \frac{d^4 l}{(2\pi)^4}\frac{1}{l^2 + m^2}.$$ **[3]**

(b) Evaluate the loop integral using a cut-off Λ and show that if $\Lambda \gg m$, the 1-loop self-energy is given by

$$\mathcal{A}_2 = -\frac{ig}{32\pi^2}\Lambda^2.$$ **[4]**

(c) Draw the 2-loop diagrams contributing to the self-energy. **[3]**

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4. Recall that under a Lorentz transformation Λ, a Dirac spinor transforms as

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta\psi^\beta\left(\Lambda^{-1}x\right),$$

where

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right], \quad S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right],$$

and Ω is real and encodes boosts and rotations.

(a) Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, show that

$$S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0.$$

[5]

(b) Show that $\bar{\psi}\psi$ is a Lorentz scalar. **[2]**

(c) Show that $\bar{\psi}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi$ is a Lorentz scalar. **[3]**

ED01/2026 **CONTINUED**

SECTION B

5. Consider a free real scalar field

$$\phi(x) = \phi^+(x) + \phi^-(x) \ , \qquad \phi^+(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}} \exp\big(-i\omega_{\vec{k}} x^0 + i\vec{k}\cdot\vec{x}\big) \ ,$$

$$\phi^-(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a^\dagger_{\vec{k}} \exp\big(+i\omega_{\vec{k}} x^0 - i\vec{k}\cdot\vec{x}\big) \ ,$$

and the Feynman propagator

$$G(y-x) = \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \exp\big(-i\omega_{\vec{k}}(y^0 - x^0) + i\vec{k}\cdot(\vec{y}-\vec{x})\big)$$
$$+ \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \exp\big(-i\omega_{\vec{k}}(x^0 - y^0) + i\vec{k}\cdot(\vec{x}-\vec{y})\big) \ ,$$

where $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ are creation and annihilation operators and $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$.

(a) Using the canonical commutation relations for $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$, and taking $y^0 > x^0$, show that

$$[\phi^+(y), \phi^-(x)] = G(y-x)\mathbf{1} \ ,$$

where $\mathbf{1}$ is the identity operator. **[5]**

(b) Define the time ordered and normal ordered products of fields. **[2]**

(c) Expand the time ordered product

$$\overleftarrow{\mathrm{T}}\big(\phi(z)\phi(y)\phi(x)\big) \ ,$$

in terms of the field $\phi(x)$ and the Heaviside step function $\theta(x^0)$, and expand the normal ordered product

$$:\phi(z)\phi(y)\phi(x): \ ,$$

in terms of $\phi^+(x)$ and $\phi^-(x)$. **[2]**

(d) Let Δ be the difference between the time ordered and normal ordered products

$$\Delta = \overleftarrow{\mathrm{T}}\big(\phi(z)\phi(y)\phi(x)\big) - :\phi(z)\phi(y)\phi(x): \ .$$

Using the identity from part (a) and taking $z^0 > y^0 > x^0$, show that

$$\Delta = \phi(z)G(y-x) + \phi(y)G(z-x) + \phi(x)G(z-y) \ .$$ **[6]**

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6. Consider the interacting theory of two real scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the action

$$S[\phi_1, \phi_2] = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m_1^2}{2}\phi_1^2 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m_2^2}{2}\phi_2^2 - \frac{\lambda}{3!}\phi_1^3\phi_2 \right) .$$

(a) Denoting the Feynman propagators for the fields ϕ_1 and ϕ_2 by $G_1(y-x)$ and $G_2(y-x)$ respectively, write down the position space Feynman rules for calculating time ordered correlation functions in this theory. **[4]**

(b) Briefly explain why vacuum bubbles do not contribute to time ordered correlation functions in the interacting theory. **[2]**

(c) Draw the Feynman diagrams contributing to the time ordered correlation functions

$$\langle \Omega | \overleftarrow{\mathrm{T}}\big(\phi_1(x_2)\phi_1(x_1)\big) | \Omega \rangle \ ,$$
$$\langle \Omega | \overleftarrow{\mathrm{T}}\big(\phi_2(x_2)\phi_2(x_1)\big) | \Omega \rangle \ ,$$
$$\langle \Omega | \overleftarrow{\mathrm{T}}\big(\phi_2(x_2)\phi_1(x_1)\big) | \Omega \rangle \ ,$$

up to and including $\mathcal{O}(\lambda^2)$, where $|\Omega\rangle$ is the interacting theory vacuum. **[7]**

(d) Using the Feynman rules, write down the expression for the last of these time ordered correlation functions

$$\langle \Omega | \overleftarrow{\mathrm{T}}\big(\phi_2(x_2)\phi_1(x_1)\big) | \Omega \rangle \ ,$$

as an expansion in λ up to and including $\mathcal{O}(\lambda^2)$. You may express your result as an integral over the Feynman propagators $G_1(y-x)$ and $G_2(y-x)$. **[2]**

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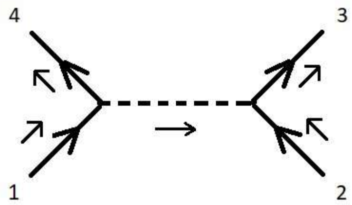
7. Consider the Yukawa theory

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}\left(i\ \partial\!\!\!/ - m\right)\psi + \lambda\phi\bar{\psi}\psi.$$

(a) Write the momentum space Feynman rules for this theory. Use dotted lines for the scalar and solid lines for the fermion. **[3]**

(b) Compute the non-relativistic limit of the following diagram:



and show that it is given by $(2m)^2\mathcal{A}_4\left(\vec{p}_1 - \vec{p}_4\right)\delta_{s_1,s_4}\delta_{s_2,s_3}$ where

$$\mathcal{A}_4\left(\vec{p}_1 - \vec{p}_4\right) = \frac{i\lambda^2}{\left(\vec{p}_1 - \vec{p}_4\right)^2 + M^2}.$$

Recall that $u_s\left(\vec{p}\right) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\xi_s \\ \sqrt{-p\cdot\sigma}\xi_s \end{pmatrix}$, where $\xi_s^\dagger\xi_{s'} = \delta_{s,s'}$. **[8]**

(c) In the Born approximation, the scattering amplitude for a non-relativistic particle in a potential V is given by

$$\mathcal{A}_4\left(\vec{p}\right) = -i\int d^3x\, V(\vec{x})e^{-i\vec{p}\cdot\vec{x}}.$$

Use this to determine the potential for the Yukawa theory. The following integral may be useful:

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{\vec{p}^2 + M^2} = \frac{1}{4\pi\left|\vec{x}\right|}e^{-M|\vec{x}|}.$$ **[4]**

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8. Consider a 0-dimensional scalar field theory with action $S[\phi]$ and let

$$e^{iW(J)} = \int_{-\infty}^{\infty} d\phi e^{iS(\phi)+iJ\phi}, \tag{8.1}$$

where

$$W(J) = \sum_{n=0}^{\infty} \frac{\kappa_n}{n!} J^n,$$

and $S(\phi)$ is defined such that the right-hand-side of (8.1) is one when $J = 0$. Moreover suppose that $\kappa_0 = \kappa_1 = 0$ and let

$$\langle \phi^m \rangle = \int_{-\infty}^{\infty} d\phi e^{iS(\phi)} \phi^m.$$

(a) Show that $\langle \phi \rangle = 0$ and express κ_2 and κ_3 in terms of $\langle \phi^2 \rangle$ and $\langle \phi^3 \rangle$. **[7]**

(b) Show that

$$e^{iS(\phi)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dJ e^{iW(J)-iJ\phi}.$$ **[3]**

(c) Show that

$$e^{iS(\phi)} = \sqrt{\frac{i}{2\pi\kappa_2}} \exp\left[i \sum_{n=3}^{\infty} \frac{\kappa_n}{n!} (i\partial_\phi)^n \right] e^{-\frac{i\phi^2}{2\kappa_2}}.$$ **[5]**
