



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH41020-WE01
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Title: Advanced Quantum Theory V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Consider the interacting theory of a complex scalar field $\Phi(x)$ with the action

$$S[\Phi, \Phi^*] = \int d^4x \left(-\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{\lambda}{4} \Phi^{*2} \Phi^2 \right).$$

- (a) Show that the action is invariant under the transformation

$$\Phi \rightarrow e^{i\alpha} \Phi, \quad \Phi^* \rightarrow e^{-i\alpha} \Phi^*, \quad (1.1)$$

where α is a constant parameter. [2]

- (b) Compute the conserved Noether current associated with this symmetry. [4]

- (c) Consider a second complex scalar field Ψ transforming as

$$\Psi \rightarrow e^{-2i\alpha} \Psi, \quad \Psi^* \rightarrow e^{2i\alpha} \Psi^*. \quad (1.2)$$

Let n_1, n_2, n_3 and n_4 be non-negative integers. Determine two linearly independent cubic interaction terms of the form

$$\sum_{\substack{n_1, n_2, n_3, n_4 \\ n_1 + n_2 + n_3 + n_4 = 3}} \alpha_{n_1, n_2, n_3, n_4} \Phi^{n_1} \Phi^{*n_2} \Psi^{n_3} \Psi^{*n_4},$$

that are both real and invariant under the transformations (1.1) and (1.2). [4]

2. Consider the free theory of a real scalar field $\phi(x)$ with the action

$$S[\phi] = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right).$$

Canonically quantizing, the normal ordered Hamiltonian, spatial momentum and number operators can be written in terms of creation and annihilation operators as

$$H = \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}, \quad \vec{P} = \int \frac{d^3k}{(2\pi)^3} \vec{k} a_{\vec{k}}^\dagger a_{\vec{k}}, \quad \mathcal{N} = \int \frac{d^3k}{(2\pi)^3} a_{\vec{k}}^\dagger a_{\vec{k}},$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$ and $\vec{k} = (k_1, k_2, k_3)$.

- (a) Write down the canonical commutation relations for the creation and annihilation operators. [1]

- (b) Define the vacuum state $|0\rangle$ of the theory and briefly explain how to construct single particle and multiparticle states. [3]

- (c) Show that

$$\int \frac{d^3k}{(2\pi)^3} f(\vec{k}) [a_{\vec{k}}^\dagger a_{\vec{k}}, a_{\vec{p}}^\dagger] = f(\vec{p}) a_{\vec{p}}^\dagger,$$

where $f(\vec{k})$ is an arbitrary function of \vec{k} . [2]

- (d) Consider the state

$$(a_{\vec{p}_1}^\dagger + a_{\vec{p}_2}^\dagger) |0\rangle.$$

Determine for which \vec{p}_1 and \vec{p}_2 this state is an eigenstate of the normal ordered Hamiltonian, spatial momentum and number operators. You should find a different result for each operator. [4]

3. Consider ϕ^4 theory in $d = 4$ spacetime dimensions:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4.$$

(a) The 1-loop correction to the self-energy is given by the following off-shell amputated diagram:

$$\mathcal{A}_2 = \text{---} \bigcirc \text{---}$$

Show that this is given by

$$\mathcal{A}_2 = -\frac{g}{2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + m^2}. \quad [3]$$

(b) Evaluate the loop integral using a cut-off Λ and show that if $\Lambda \gg m$, the 1-loop self-energy is given by

$$\mathcal{A}_2 = -\frac{ig}{32\pi^2}\Lambda^2. \quad [4]$$

(c) Draw the 2-loop diagrams contributing to the self-energy. [3]

4. Recall that under a Lorentz transformation Λ , a Dirac spinor transforms as

$$\psi^\alpha(x) \rightarrow S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x),$$

where

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right], \quad S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma],$$

and Ω is real and encodes boosts and rotations.

(a) Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, show that

$$S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1} \gamma^0. \quad [5]$$

(b) Show that $\bar{\psi}\psi$ is a Lorentz scalar. [2]

(c) Show that $\bar{\psi}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi$ is a Lorentz scalar. [3]

SECTION B

5. Consider a free real scalar field

$$\phi(x) = \phi^+(x) + \phi^-(x), \quad \phi^+(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}} \exp(-i\omega_{\vec{k}}x^0 + i\vec{k} \cdot \vec{x}),$$

$$\phi^-(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}}^\dagger \exp(+i\omega_{\vec{k}}x^0 - i\vec{k} \cdot \vec{x}),$$

and the Feynman propagator

$$G(y-x) = \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \exp(-i\omega_{\vec{k}}(y^0 - x^0) + i\vec{k} \cdot (\vec{y} - \vec{x}))$$

$$+ \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \exp(-i\omega_{\vec{k}}(x^0 - y^0) + i\vec{k} \cdot (\vec{x} - \vec{y})),$$

where $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are creation and annihilation operators and $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$.

- (a) Using the canonical commutation relations for $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$, and taking $y^0 > x^0$, show that

$$[\phi^+(y), \phi^-(x)] = G(y-x)\mathbf{1},$$

where $\mathbf{1}$ is the identity operator. [5]

- (b) Define the time ordered and normal ordered products of fields. [2]

- (c) Expand the time ordered product

$$\overleftarrow{\text{T}}(\phi(z)\phi(y)\phi(x)),$$

in terms of the field $\phi(x)$ and the Heaviside step function $\theta(x^0)$, and expand the normal ordered product

$$:\phi(z)\phi(y)\phi(x):,$$

in terms of $\phi^+(x)$ and $\phi^-(x)$. [2]

- (d) Let Δ be the difference between the time ordered and normal ordered products

$$\Delta = \overleftarrow{\text{T}}(\phi(z)\phi(y)\phi(x)) - :\phi(z)\phi(y)\phi(x):.$$

Using the identity from part (a) and taking $z^0 > y^0 > x^0$, show that

$$\Delta = \phi(z)G(y-x) + \phi(y)G(z-x) + \phi(x)G(z-y). \quad [6]$$

6. Consider the interacting theory of two real scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the action

$$S[\phi_1, \phi_2] = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{m_1^2}{2} \phi_1^2 - \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{m_2^2}{2} \phi_2^2 - \frac{\lambda}{3!} \phi_1^3 \phi_2 \right).$$

- (a) Denoting the Feynman propagators for the fields ϕ_1 and ϕ_2 by $G_1(y-x)$ and $G_2(y-x)$ respectively, write down the position space Feynman rules for calculating time ordered correlation functions in this theory. [4]
- (b) Briefly explain why vacuum bubbles do not contribute to time ordered correlation functions in the interacting theory. [2]
- (c) Draw the Feynman diagrams contributing to the time ordered correlation functions

$$\begin{aligned} &\langle \Omega | \overleftarrow{\text{T}}(\phi_1(x_2) \phi_1(x_1)) | \Omega \rangle, \\ &\langle \Omega | \overleftarrow{\text{T}}(\phi_2(x_2) \phi_2(x_1)) | \Omega \rangle, \\ &\langle \Omega | \overleftarrow{\text{T}}(\phi_2(x_2) \phi_1(x_1)) | \Omega \rangle, \end{aligned}$$

up to and including $\mathcal{O}(\lambda^2)$, where $|\Omega\rangle$ is the interacting theory vacuum. [7]

- (d) Using the Feynman rules, write down the expression for the last of these time ordered correlation functions

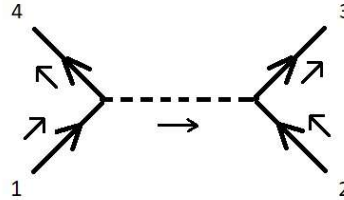
$$\langle \Omega | \overleftarrow{\text{T}}(\phi_2(x_2) \phi_1(x_1)) | \Omega \rangle,$$

as an expansion in λ up to and including $\mathcal{O}(\lambda^2)$. You may express your result as an integral over the Feynman propagators $G_1(y-x)$ and $G_2(y-x)$. [2]

7. Consider the Yukawa theory

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi + \lambda\phi\bar{\psi}\psi.$$

- (a) Write the momentum space Feynman rules for this theory. Use dotted lines for the scalar and solid lines for the fermion. [3]
- (b) Compute the non-relativistic limit of the following diagram:



and show that it is given by $(2m)^2\mathcal{A}_4(\vec{p}_1 - \vec{p}_4)\delta_{s_1,s_4}\delta_{s_2,s_3}$ where

$$\mathcal{A}_4(\vec{p}_1 - \vec{p}_4) = \frac{i\lambda^2}{(\vec{p}_1 - \vec{p}_4)^2 + M^2}.$$

Recall that $u_s(\vec{p}) = \begin{pmatrix} \sqrt{-p \cdot \sigma} \xi_s \\ \sqrt{-p \cdot \sigma} \xi_s \end{pmatrix}$, where $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$. [8]

- (c) In the Born approximation, the scattering amplitude for a non-relativistic particle in a potential V is given by

$$\mathcal{A}_4(\vec{p}) = -i \int d^3x V(\vec{x}) e^{-i\vec{p}\cdot\vec{x}}.$$

Use this to determine the potential for the Yukawa theory. The following integral may be useful:

$$\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2 + M^2} = \frac{1}{4\pi|\vec{x}|} e^{-M|\vec{x}|}. \quad [4]$$

8. Consider a 0-dimensional scalar field theory with action $S[\phi]$ and let

$$e^{iW(J)} = \int_{-\infty}^{\infty} d\phi e^{iS(\phi)+iJ\phi}, \quad (8.1)$$

where

$$W(J) = \sum_{n=0}^{\infty} \frac{\kappa_n}{n!} J^n,$$

and $S(\phi)$ is defined such that the right-hand-side of (8.1) is one when $J = 0$. Moreover suppose that $\kappa_0 = \kappa_1 = 0$ and let

$$\langle \phi^m \rangle = \int_{-\infty}^{\infty} d\phi e^{iS(\phi)} \phi^m.$$

(a) Show that $\langle \phi \rangle = 0$ and express κ_2 and κ_3 in terms of $\langle \phi^2 \rangle$ and $\langle \phi^3 \rangle$. [7]

(b) Show that

$$e^{iS(\phi)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dJ e^{iW(J)-iJ\phi}. \quad [3]$$

(c) Show that

$$e^{iS(\phi)} = \sqrt{\frac{i}{2\pi\kappa_2}} \exp \left[i \sum_{n=3}^{\infty} \frac{\kappa_n}{n!} (i\partial_\phi)^n \right] e^{-\frac{i\phi^2}{2\kappa_2}}. \quad [5]$$