Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH41120-WE01

Title:
Algebraic Topology V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. This question is about short exact sequences of abelian groups.

(a) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that $A \cong D$ and $B \cong E$, but $C \ncong F$. **[3]**

(b) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that $A \cong D$ and $C \cong F$, but $B \ncong E$. **[2]**

(c) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that $B \cong E$ and $C \cong F$, but $A \ncong D$. **[5]**

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2. Let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere inside 3-dimensional space. Compute the homology groups of the quotient space $H_n(\mathbb{R}^3/S^2)$ for all $n \geq 0$. **[10]**

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3. For $n \in \mathbb{Z}$, define $C_n = \mathbb{Z}/18\mathbb{Z}$ and let $\partial_n \colon C_n \to C_{n-1}$ be given by $\partial_n(\overline{k}) = \overline{6k}$.

(a) Show that (C_*, ∂) is a chain complex and calculate $H_n(C_*)$. **[3]**

(b) Calculate $H^n(\mathrm{Hom}(C_*, \mathbb{Z}/4\mathbb{Z}))$ for all $n \in \mathbb{Z}$. **[3]**

(c) Calculate $H^n(\mathrm{Hom}(C_*, \mathbb{Z}/9\mathbb{Z}))$ for all $n \in \mathbb{Z}$. **[4]**

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4. Let M be a compact, connected manifold of dimension m, possibly with boundary.

(a) State the definition of M being $\mathbb{Z}$-orientable. **[3]**

(b) Assume that $\partial M \neq \emptyset$. Show that $H_m(M) = 0$. You may assume that M admits a triangulation. State any results from the lectures that you use. **[7]**

SECTION B

5. (a) Using a Mayer-Vietoris sequence or otherwise, compute the homology groups of the space Y given as a subspace of $\mathbb{R}^3$ as the union of the unit 2-sphere together with the unit coordinate axes:

$$\begin{aligned} Y =& \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\} \\ &\cup \{(x,0,0) \in \mathbb{R}^3 : -1 \leq x \leq 1\} \\ &\cup \{(0,y,0) \in \mathbb{R}^3 : -1 \leq y \leq 1\} \\ &\cup \{(0,0,z) \in \mathbb{R}^3 : -1 \leq z \leq 1\}. \end{aligned}$$ **[6]**

(b) Assuming that the homology groups $H_n(X)$ are free abelian for each $n \in \mathbb{Z}$, show using a Mayer-Vietoris long exact sequence that

$$H_n(S^1 \times X) \cong H_{n-1}(X) \oplus H_n(X)$$

for all $n \in \mathbb{Z}$. **[9]**

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6. Suppose that (A_*, d_*^A), (B_*, d_*^B), and (C_*, d_*^C) are chain complexes of free abelian groups. Suppose further that

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

is a short exact sequence of these chain complexes (where f_* and g_* are chain maps).

(a) In such a situation there is a *connecting homomorphism*

$$\delta_n \colon H_n(C_*) \longrightarrow H_{n-1}(A_*)$$

which fits into a long exact sequence of homology groups by the snake lemma. If $[c] \in \ker(d_n^C)/\mathrm{im}(d_{n-1}^C) = H_n(C_*)$, explain how to construct the element $\delta_n([c]) \in H_{n-1}(A_*)$ (you need not show that this element is well-defined, and you need not derive any of the properties of the long exact sequence). **[5]**

(b) Give an example of a short exact sequence of chain complexes of free abelian groups

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

in which $H_1(C_*) = H_0(A_*) = \mathbb{Z}/4\mathbb{Z}$ and for which the connecting homomorphism satisfies $\delta_1(m) = 2m$ for all $m \in \mathbb{Z}/4\mathbb{Z}$. **[10]**

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7. Recall that the Universal Coefficient Theorem gives a short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), \mathbb{K}) \longrightarrow H^n(X;\mathbb{K}) \xrightarrow{\Phi} \operatorname{Hom}(H_n(X), \mathbb{K}) \longrightarrow 0$$

where X is a topological space, $\mathbb{K}$ a commutative ring, and Φ the map induced by the Kronecker product.

(a) Explain what it means that this short exact sequence is *natural*. **[4]**

(b) The Universal Coefficient Theorem also states that the above short exact sequence splits. Explain what this means. **[2]**

(c) Consider the quotient map $f\colon \mathbb{R}\mathbf{P}^2 \to \mathbb{R}\mathbf{P}^2/\mathbb{R}\mathbf{P}^1$. Show that

$$f^*\colon H^2(\mathbb{R}\mathbf{P}^2/\mathbb{R}\mathbf{P}^1; \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{R}\mathbf{P}^2; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism. Here $\mathbb{R}\mathbf{P}^n$ is projective n-space. **[4]**

(d) By using part (c), or otherwise, provide an example to show that the splitting of the above sequence is not natural in general. **[5]**

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8. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit n-sphere inside $(n+1)$-dimensional space.

(a) Describe the cohomology ring structure of $S^2 \times S^2$ without proof. **[4]**

(b) Give an example of a map $f\colon S^2 \times S^2 \to S^4$ which induces an isomorphism

$$f^*\colon H^4(S^4;\mathbb{Z}) \to H^4(S^2 \times S^2;\mathbb{Z}),$$

and justify your statement. **[6]**

(c) Let $g\colon S^4 \to S^2 \times S^2$ be any map. Show that

$$g^*\colon H^4(S^2 \times S^2;\mathbb{Z}) \to H^4(S^4;\mathbb{Z})$$

is the 0-map. **[5]**
