



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH41220-WE01
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Title: Analysis V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Let μ^* denote the Lebesgue outer measure. Let $C \subset [0, 1]$ denote the middle third Cantor set.

(a) Prove that $\mu^*(C) = 0$.

Remember to state all the properties of μ^* that you use.

[You may use the fact that for any $n \in \mathbb{N}$, we can cover C by 2^n intervals of length 3^{-n}].

[5]

(b) Let $A = [0, 1] \setminus C$. What is the Lebesgue outer measure of A ?

Give a full justification of your response stating all the properties of μ^* that you use.

[5]

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and $1 \leq p < \infty$.

(a) State the definition of $L^p(E)$ and its norm $\|\cdot\|_{L^p}$.

[4]

(b) Let $f : E \rightarrow \mathbb{R}$ be Lebesgue measurable. Prove that $\int_E |f|^p = 0$ if and only if f vanishes almost everywhere in E .

Note that it is **not** sufficient to state that the L^p -norm is a norm.

[6]

3. (a) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let p, q be conjugate exponents. State Hölder's Inequality for $f \in L^p(E)$, $g \in L^q(E)$.

[1]

(b) Prove or disprove that $L^2([0, 1]) \subset L^1([0, 1])$.

[3]

(c) Define $T : L^2[0, 1] \rightarrow \mathbb{R}$ by

$$T(F) := \int_{[0, \frac{1}{2}]} F.$$

Prove that T is a bounded linear functional.

[6]

SECTION B

4. (a) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_n(x) = n \cdot \mathbb{1}_{(0,1/n)}(x) = \begin{cases} n, & \text{if } x \in (0, 1/n), \\ 0, & \text{otherwise,} \end{cases}$$

and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 0$. Then $(f_n)_{n=1}^\infty$ converges pointwise to f as $n \rightarrow \infty$. Does the Monotone Convergence Theorem apply to $(f_n)_{n=1}^\infty$? Does Fatou's Lemma apply to $(f_n)_{n=1}^\infty$? Justify your responses and compute the integrals $\int f_n$, $\int f$. [7]

- (b) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let $(g_n)_{n=1}^\infty$ be an increasing sequence of extended real-valued nonnegative Lebesgue measurable functions on E . Suppose that the sequence of real numbers $(\int_E g_n)_{n=1}^\infty$ is bounded, i.e., there exists $M > 0$ such that $|\int_E g_n| \leq M$ for all $n \in \mathbb{N}$.

Prove that there exists a Lebesgue measurable function g such that the sequence $(g_n)_{n=1}^\infty$ converges pointwise on E to g and hence deduce that

$$\lim_{n \rightarrow \infty} \int_E g_n = \int_E g < \infty.$$

In addition, prove that g is finite almost everywhere on E .

Remember to state the names of any results from lectures that you use. [8]

5. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set.

(a) (i) State what it means for an extended real-valued function $h : E \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ to be Lebesgue measurable. [2]

(ii) Let $h : E \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be an extended real-valued Lebesgue measurable function. For $n \in \mathbb{N}$, let $h_n : E \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be defined by

$$h_n(x) = \begin{cases} h(x), & \text{if } |h(x)| \leq n, \\ n, & \text{if } h(x) > n, \\ -n, & \text{if } h(x) < -n. \end{cases}$$

Prove that h_n is Lebesgue measurable for all $n \in \mathbb{N}$. [5]

(b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of integrable functions on E that converges pointwise almost everywhere on E to f . Suppose that there is a sequence $(g_n)_{n=1}^{\infty}$ of nonnegative integrable functions on E that converges pointwise almost everywhere on E to g and satisfies

$$\lim_{n \rightarrow \infty} \int_E g_n = \int_E g < \infty,$$

and

$$|f_n| \leq g_n \text{ on } E \text{ for all } n \in \mathbb{N}.$$

Prove that

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f. \quad [8]$$

6. (a) Let $(X, \|\cdot\|)$ be a normed linear space. State what it means for $(X, \|\cdot\|)$ to be a Banach space. [2]

(b) Let $C[0, 2]$ be the linear space of continuous functions $f : [0, 2] \rightarrow \mathbb{R}$. Consider $C[0, 2]$ equipped with the norm $\|f\|_{L^1} = \int_{[0,2]} |f|$. For $n \in \mathbb{N}$, let $f_n : [0, 2] \rightarrow \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{4}), \\ n(x - \frac{1}{4}), & x \in [\frac{1}{4}, \frac{1}{4} + \frac{1}{n}), \\ 1, & x \in [\frac{1}{4} + \frac{1}{n}, 2]. \end{cases}$$

(i) Prove that $f_n \in C[0, 2]$ and that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(C[0, 2], \|\cdot\|_{L^1})$. [5]

(ii) Is $(C[0, 2], \|\cdot\|_{L^1})$ a Banach space? Give a full justification of your response. [8]

7. (a) Let $M \subset \mathcal{H}$ be a closed subspace of a Hilbert space \mathcal{H} with $M \neq \mathcal{H}$. Denote by $P : \mathcal{H} \rightarrow M$ the orthogonal projection of \mathcal{H} onto M , i.e., if $x \in \mathcal{H}$, then $Px \in M$ is the unique closest point to x in M .

Let $x \in \mathcal{H}$. Suppose that $y \in M$ is such that $(x - y) \perp M$. Prove that $y = Px$. [5]

- (b) Recall that $\{e^{ikx} : k \in \mathbb{Z}\}$ is an orthonormal set in the Hilbert space $L^2[-\pi, \pi]$ equipped with the inner product for $f, g \in L^2[-\pi, \pi]$:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f \bar{g}, \quad f, g \in L^2[-\pi, \pi].$$

Recall that the Fourier coefficients of $f \in L^2[-\pi, \pi]$ are defined as

$$a_k(f) = \langle f, e^{iky} \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f(y) e^{-iky}.$$

Let $n \in \mathbb{N}$. By using the fact that

$$0 \leq \frac{1}{2\pi} \int_{[-\pi, \pi]} \left| f - \sum_{k=-n}^n a_k(f) e^{ikx} \right|^2,$$

prove that

$$\sum_{k \in \mathbb{Z}} |a_k(f)|^2 \leq \int |f|^2. \quad [10]$$

SECTION C

8. Let (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) be measure spaces.

(a) State the Product Measure Theorem. [2]

(b) State Tonelli's Theorem. [3]

- (c) Let $([0, 1], \mathcal{B}([0, 1]), \mu)$ be the σ -finite measure space given by the Borel σ -algebra $\mathcal{B}([0, 1])$ and the Lebesgue measure μ . Let π be the measure given in the Product Measure Theorem for $(X, \mathcal{X}, \mu) = (Y, \mathcal{Y}, \nu) = ([0, 1], \mathcal{B}([0, 1]), \mu)$. Consider the measurable function $\psi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by

$$\psi(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1], y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate, with justification, the integral $\int_{[0,1] \times [0,1]} \psi d\pi$. [5]