



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH41320-WE01
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Title: Riemannian Geometry V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Prove or disprove the following assertions:

(a) The subset of \mathbb{R}^2 given by

$$M = \{ (x, y) \in \mathbb{R}^2 : |y| = |x| \}$$

is a smooth embedded submanifold of \mathbb{R}^2 . [5]

(b) The subset of \mathbb{R}^4 given by

$$M = \{ (w, x, y, z) \in \mathbb{R}^4 : w + e^x + 3y + 2z^4 = 0 \}$$

is a smooth 3-dimensional submanifold of \mathbb{R}^4 . [5]

2. (a) Prove or disprove the following assertion: *there exists a Riemannian metric g on the real projective plane $\mathbb{R}P^2$ such that, for some chart $(U, (x^1, x^2))$, the local expression of g on U is given by*

$$(g_{ij}) = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}. \quad [4]$$

(b) State the definition of a diffeomorphism between smooth manifolds. [2]

(c) Let M and N be 3-smooth manifolds and let $f: M \rightarrow N$ be a smooth function whose differential in local coordinates around $p \in M$ and $f(p) \in N$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Can f be a diffeomorphism? Justify your answer. [4]

3. Let ∇^1 and ∇^2 be affine connections on a smooth manifold M .

(a) Prove or disprove the following assertion: *the map*

$$\begin{aligned} \nabla: \mathcal{X}(M) \times \mathcal{X}(M) &\rightarrow \mathcal{X}(M) \\ (X, Y) &\mapsto \nabla_X^1 Y + \nabla_X^2 Y \end{aligned}$$

defines an affine connection on M , where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M .

[5]

(b) Let T^1 and T^2 denote the torsion tensors of ∇^1 and ∇^2 , respectively, and suppose that ∇^1 and ∇^2 are compatible with a Riemannian metric g on M . The difference tensor for ∇^1 and ∇^2 is given by

$$\begin{aligned} A: \mathcal{X}(M) \times \mathcal{X}(M) &\rightarrow \mathcal{X}(M) \\ (X, Y) &\mapsto \nabla_X^1 Y - \nabla_X^2 Y. \end{aligned}$$

Show that the difference tensor A satisfies

- (i) $A(X, Y) - A(Y, X) = T^1(X, Y) - T^2(X, Y)$, and
- (ii) $g(A(X, Y), Z) = -g(Y, A(X, Z))$

for all $X, Y, Z \in \mathcal{X}(M)$.

[5]

4. Prove or disprove the following assertions. You may use, without proof, any relevant results from the lectures.

(a) *If (M, g) is a complete, path-connected Riemannian manifold with Riemannian distance d_g and $p, q \in M$ are points in M , then there exists a piecewise-smooth curve $c: [a, b] \rightarrow M$ with $c(a) = p$, $c(b) = q$ and length $L(c) = d_g(p, q)$.*

[2]

(b) *The n -dimensional sphere S^n admits a complete Riemannian metric with sectional curvature $K \leq 0$.*

[4]

(c) *There exist positive integers n and k such that the smooth product manifold $S^n \times \mathbb{R}^k$ admits a complete Riemannian metric with sectional curvature $K \geq 3$.*

[4]

SECTION B

5. Let

$$X(x, y, z) = z \frac{\partial}{\partial x} - 2z \frac{\partial}{\partial y} + (2y - x) \frac{\partial}{\partial z},$$
$$Y(x, y, z) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},$$

be two smooth vector fields on \mathbb{R}^3 equipped with the usual chart given by the identity map.

- (a) Compute the Lie bracket $[X, Y]$. [5]
 - (b) Let $S^2(1) \subset \mathbb{R}^3$ be the unit round sphere centred at the origin. Show that the restrictions of the vector fields X and Y to $S^2(1)$ are vector fields on $S^2(1)$, i.e. $X(p)$ and $Y(p)$ are tangent to $S^2(1)$ for any $p \in S^2(1)$. [6]
 - (c) Show that the restriction of $[X, Y]$ to $S^2(1)$ is also a vector field on $S^2(1)$. [4]
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- 6. (a) State the definition of a Lie group. [4]
 - (b) Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices. Show that the general linear group $\text{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ is a Lie group. [6]
 - (c) Let (M, g) be a Riemannian manifold and let X be a smooth vector field on M . Show that $p \mapsto g_p(X(p), \cdot)$ is a smooth differential form on M . [5]
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7. Let (M_1, g_1) and (M_2, g_2) be path-connected Riemannian manifolds and suppose that there exists an isometry $f: (M_1, g_1) \rightarrow (M_2, g_2)$. Let $c: [a, b] \rightarrow M_1$ be a piecewise smooth curve.

- (a) Show that

$$L(c) = L(f \circ c),$$

where $L(c)$ and $L(f \circ c)$ are the lengths of the curves c and $f \circ c$ in (M_1, g_1) and (M_2, g_2) respectively. [2]

- (b) Show that

$$d_{g_1}(c(a), c(b)) = d_{g_2}(f(c(a)), f(c(b))),$$

where d_{g_1} and d_{g_2} denote the Riemannian distances in (M_1, g_1) and (M_2, g_2) respectively. [4]

- (c) If $c: [a, b] \rightarrow M_1$ is a minimising geodesic segment, show that the curve $f \circ c: [a, b] \rightarrow M_2$ is a minimising geodesic segment and is given by

$$f(c(t)) = \exp_{f(c(a))}(t Df_{f(c(a))}(c'(a))), \quad t \in [a, b]. \quad [4]$$

- (d) Let $I \in \mathbb{R}$ be an open interval and let $\gamma: I \rightarrow M_1$ be a geodesic in (M_1, g_1) . Show that $f \circ \gamma: I \rightarrow M_2$ is a geodesic in (M_2, g_2) . [2]

- (e) Show that (M_1, g_1) is geodesically complete if and only if (M_2, g_2) is geodesically complete. [3]

8. Let (M, g) be a three-dimensional Riemannian manifold with Levi-Civita connection ∇ . Suppose that there exist smooth vector fields $X_1, X_2, X_3 \in \mathcal{X}(M)$ on M such that

$$g(X_i, X_j) = \delta_{ij},$$

for all $i, j \in \{1, 2, 3\}$, where $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$. Suppose, furthermore, that

$$\nabla_{X_1} X_2 = X_3, \quad \nabla_{X_2} X_3 = X_1, \quad \nabla_{X_3} X_1 = X_2 \quad \text{and} \quad \nabla_{X_i} X_j = -\nabla_{X_j} X_i,$$

for all $i, j \in \{1, 2, 3\}$.

- (a) Show that the Lie bracket satisfies

$$[X_i, X_j] = \begin{cases} 2X_k, & \text{if } (i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}, \\ -2X_k, & \text{if } (i, j, k) \in \{(2, 1, 3), (1, 3, 2), (3, 2, 1)\}. \end{cases}$$

Note that this implies that the vector fields X_1, X_2, X_3 are not coordinate vector fields. [3]

- (b) Show that the curvature tensor R of (M, g) satisfies

$$R(X_i, X_j)X_k = \begin{cases} X_i, & \text{if } i, j, k \in \{1, 2, 3\} \text{ with } i \neq j, j = k, \\ -X_j, & \text{if } i, j, k \in \{1, 2, 3\} \text{ with } i \neq j, i = k, \\ 0, & \text{otherwise.} \end{cases} \quad [5]$$

- (c) Show that the sectional curvature K satisfies

$$K(X_i, X_j) = 1,$$

for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. [2]

- (d) Show that (M, g) has constant sectional curvature; that is, show that

$$K(v, w) = 1,$$

for all $p \in M$ and all $v, w \in T_p M$ with $\|v\| = 1$, $\|w\| = 1$ and $g_p(v, w) = 0$. [5]
