Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH41420-WE01

Title:
Solitons V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider the ball and box model. At time $t = 0$ there are balls in boxes 0, 1 and 4.

(a) Evolving the system by the ball and box rule, where are the balls at times $t = 1$, 2 and 3? **[4]**

(b) What are the phase shifts of the two solitons present in this system? **[3]**

(c) Where is the slower soliton at times $t = 25$ and $t = 100$? **[3]**

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2. (a) Define the Initial Value Problem for a partial differential equation involving a function $u(x,t)$ which vanishes as $x \to \pm\infty$. Describe the method for solving the Initial Value Problem for linear partial differential equations. You may use diagrams as part of your answer so long as you add sufficient explanations. **[5]**

(b) Now describe the Inverse Scattering Method for solving the Initial Value Problem for the KdV equation. Outline the scattering data and point out an important property which allows this method to proceed. Again, you may use diagrams as part of your answer so long as you add sufficient explanation. **[5]**

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3. (a) Let $u(x)$ be a function of the real coordinate x that satisfies the boundary conditions $u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ as $x \to \pm\infty$, and $F[u]$ be the functional

$$F[u] = \int_{-\infty}^{+\infty} dx \; f(u, u_x, u_{xx}) \; .$$

Define the functional derivative $\delta F[u]/\delta u$, and derive an expression for this functional derivative in terms of $\partial f/\partial u$, $\partial f/\partial u_x$ and $\partial f/\partial u_{xx}$. **[5]**

(b) Now take $f(u, u_x, u_{xx}) = 2uu_x^2 + u^3 + u^2 u_{xx}$, and compute $\delta F[u]/\delta u$. Is there a simple explanation for the result you obtained? **[5]**

SECTION B

4. (a) Prove that

$$Q_1 = \int_{-\infty}^{\infty} u \, dx \quad \text{and} \quad Q_2 = \int_{-\infty}^{\infty} u^2 \, dx$$

are both conserved quantities for the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, with boundary conditions that u, u_x and u_{xx} tend to zero as $|x| \to \infty$. **[5]**

(b) Evaluate Q_1 and Q_2 for a single KdV soliton with velocity $v > 0$, given by

$$u(x,t) = \frac{v}{2} \operatorname{sech}^2 \left[\frac{\sqrt{v}}{2}(x - x_0 - vt) \right] .$$

Here, x_0 is the location of the soliton at $t = 0$. You may use the definite integrals $\int_{-\infty}^{\infty} \operatorname{sech}^2(x)\, dx = 2$ and $\int_{-\infty}^{\infty} \operatorname{sech}^4(x)\, dx = 4/3$ without proof. **[5]**

(c) The initial condition

$$u(x,0) = \frac{3}{2} \operatorname{sech}^2(x/2)$$

evolves as $t \to \infty$ into a pair of single solitons, one of which has velocity $v_1 = 1$. Use the conservation of Q_1 to find the velocity v_2 of the other soliton. **[5]**

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5. (a) Consider the pair of equations

$$v_x = -uv \, , \quad v_t = u^2 v - u_x v \, .$$

Show that these relations give a Bäcklund transformation between the heat equation $v_t - v_{xx} = 0$ for v, and an equation for u which you should find. **[10]**

(b) Verify that $u(x,t) = c$, with c a constant, is a solution of the equation for u that you found in part (a), and find the corresponding solution of the heat equation for v. **[5]**

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6. (a) Determine the general solution of the Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

for a potential $V(x) = a^2\theta(x)$ where $E > a^2 > 0$ and $\theta(x)$ is the unit step function, which equals 1 for positive x and 0 for negative x. You may state the matching (or junction) conditions for ψ and ψ' at $x = 0$ without proof, but you should provide a brief justification for them. **[7]**

(b) In this case, the potential does not vanish as $x \to \infty$ and so the wave number of the transmitted wave is different from that of the incoming and reflected waves. Because of this we have to carefully normalise the contributions from incoming, reflected and transmitted waves, with a $1/\sqrt{|k|}$ factor for each, where k is the wave number. For example, at $x \to -\infty$ we normalise the incoming waves from the left as

$$\frac{1}{\sqrt{k}} e^{ikx} \ .$$

Determine the reflection coefficient R and transmission coefficient T of these properly normalised waves, and show that the reflection and transmission coefficients obey the usual $|R|^2 + |T|^2 = 1$ relation. **[8]**

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7. Let $u = u(x^+, x^-)$ be a smooth function of the coordinates $x^\pm := \frac{1}{2}(t \pm x)$, and

$$P = \lambda\frac{\sigma_3}{2} - iu_+\frac{\sigma_2}{2} \ , \qquad M = -\lambda^{-1}\left(\cos(u)\frac{\sigma_3}{2} + \sin(u)\frac{\sigma_1}{2}\right) \ ,$$

where $\lambda \neq 0$ is a constant, the notation $u_\pm := \partial u/\partial x^\pm = \partial_\pm u$ has been used, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

(a) Calculate the commutator $[\sigma_a, \sigma_b]$ of all pairs of Pauli matrices. **[5]**

(b) Find the consistency condition for the system of equations

$$\partial_+\Psi = P\Psi \ , \qquad \partial_-\Psi = M\Psi$$

for a two-dimensional column vector $\Psi(x^+, x^-)$ of smooth functions of $x^\pm$, with P and M as above. **[10]**

SECTION C

8. (a) A model is defined on the half line $-\infty < x < 0$. For $x < 0$ the field $u(x,t)$ satisfies the ϕ^4 equation

$$u_{tt} - u_{xx} + 2u(u^2 - 1) = 0\,.$$

As $x \to -\infty$ the boundary conditions are $u \to -1$, u_t, $u_x \to 0$ while at $x = 0$ the field satisfies a Robin boundary condition, with constant K:

$$\frac{1}{2K} u_x(0,t) + u(0,t) = 0\,.$$

Show that the half-line energy

$$E_K[u] = \int_{-\infty}^{0} \mathcal{E}\, dx + K\, u(0,t)^2$$

is conserved, where

$$\mathcal{E} = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}(u^2 - 1)^2\,.$$

(You might start by showing that $\partial\mathcal{E}/\partial t + \partial j/\partial x = 0$ for $x < 0$, where j is some other density which you should determine.) **[6]**

(b) The model is now further restricted to the interval $-1 < x < 0$, and a second Robin boundary condition, this time with constant L, is imposed at $x = -1$:

$$\frac{1}{2L} u_x(-1,t) + u(-1,t) = 0\,.$$

Find a value of the constant A for which the interval energy

$$E_{A,K}[u] = \int_{-1}^{0} \mathcal{E}\, dx + A\, u(-1,t)^2 + K\, u(0,t)^2$$

is conserved, where $\mathcal{E}$ is as defined in part (a). Explain your answer. **[4]**
