Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2026 | MATH4151-WE01 |

**Title:** Topics in Algebra and Geometry IV

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| Time: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.<br>Write your answer in the white-covered answer booklet with barcodes.<br>Begin your answer to each question on a new page. |

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| **Revision:** | |

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## SECTION A

**1.** In this problem, we will construct a basis of cusp forms of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$. In the sequel, we write $k = 12\ell + a$, and assume that $\ell \geq 1$ and $a \in \{0, 4, 6, 8, 10, 14\}$.

(a) Use the dimension formulae to show that $\dim(S_k(\mathrm{SL}_2(\mathbb{Z}))) = \ell$. **[3]**

(b) For each $1 \leq m \leq \ell$, define

$$e_{k,m}(\tau) := \Delta(\tau)^{\ell} E_a(\tau) j(\tau)^{\ell - m},$$

where $j(\tau)$ is the modular $j$-function.
Show that $e_{k,m}$ is a holomorphic cusp form of weight $k$ that has a zero of order $m$ at infinity.
Here we recall that when $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$,

$$\Delta(\tau) = q + \sum_{n=2}^{\infty} \tau(n) q^n, \quad j(\tau) = q^{-1} + 744 + \cdots, \quad E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $B_k$ is the $k$th Bernoulli number (and $E_0$ is the constant function 1). **[3]**

(c) Show that the set $\{e_{k,m}\}_{1 \leq m \leq \ell}$ forms a basis for $S_k(\mathrm{SL}_2(\mathbb{Z}))$. **[4]**

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**2.** Recall that for $N \geq 1$, we define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Let $p$ be a prime and let $k \geq 2$ be an even integer. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be a modular form of weight $k$ with $q$-expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\mathrm{SL}_2(\mathbb{Z})).$$

Now, define

$$\tilde{f}(\tau) := \sum_{m=0}^{\infty} a_{pm} q^m.$$

(a) Show that $\tilde{f}(\tau) = T_p f(\tau) - p^{k-1} f(p\tau)$, where $T_p$ is the $p$th Hecke operator. **[3]**

(b) Prove that $f_p(\tau) := f(p\tau) \in M_k(\Gamma_0(p))$. **[4]**

(c) Show that $\tilde{f} \in M_k(\Gamma_0(p))$. **[3]**

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**3.** We say that an arithmetic function $g(n)$ is **additive** if, whenever $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, we have

$$g(ab) = g(a) + g(b).$$

(a) Recall that $\omega(n) := |\{p \text{ prime} : \; p|n\}|$. Show that $\omega(n)$ is an additive function. **[3]**

(b) Show that if $f(n)$ is a multiplicative function taking positive real values then $\log f(n)$ is an additive function. **[3]**

(c) Suppose that $g : \mathbb{N} \to \mathbb{C}$ is an additive function. Show that whenever $n > 1$ has prime factorisation $n = p_1^{a_1} \cdots p_k^{a_k}$, we have

$$g(n) = g(p_1^{a_1}) + \cdots + g(p_k^{a_k}).$$ **[4]**

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**4.** Recall that a positive integer $n$ is called **squarefree** if, for any prime $p$ that divides $n$, we have $p^2 \nmid n$.

(a) Show that

$$|\{n \leq x : \; n \text{ is squarefree}\}| = \sum_{n \leq x} \mu^2(n),$$

where $\mu(n)$ denotes the Möbius function. **[2]**

(b) Using the identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d), \quad n \geq 1,$$

prove the estimate

$$|\{n \leq x : \; n \text{ is squarefree}\}| = x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} + O(\sqrt{x}).$$ **[4]**

(c) Recall that for $\operatorname{Re}(s) > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Use this to prove that

$$\sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\frac{1}{\sqrt{x}}\right)$$

and hence deduce that

$$\lim_{x \to \infty} \frac{1}{x} |\{n \leq x : \; n \text{ is squarefree}\}| = \frac{1}{\zeta(2)}.$$ **[4]**

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## SECTION B

**5.** For $\mathrm{Im}(\tau) > 0$ let $L = [1, \tau]$ be a lattice. We define the **Weierstrass $\zeta$ function** of $L$ by

$$\zeta_L(z) = \frac{1}{z} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left( \frac{1}{z + m\tau + n} - \frac{1}{m\tau + n} + \frac{z}{(m\tau + n)^2} \right).$$

(a) Prove that

$$\zeta_L(z) = \frac{1}{z} + z^2 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau + n)^2(z + m\tau + n)}.$$ **[2]**

(b) Show that $\zeta_L(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}\setminus L$, and therefore defines a meromorphic function on $\mathbb{C}$. You may use without proof that there is a constant $C = C(\tau) > 0$ such that

$$|m\tau + n| \geq C(|m| + |n|) \text{ for all } m, n \in \mathbb{Z}.$$

(Hint: Given a compact subset $K$ with $|z| \leq R$ for all $z \in K$, it suffices to consider the contribution from $\max\{|m|, |n|\} \geq 2R/C$.) **[5]**

Next, we establish some properties of $\zeta_L$, through its relationship with the Weierstrass $\wp$-function $\wp_L$ of $L$.

(c) Show that $\zeta_L'(z) = -\wp_L(z)$, and that $\zeta_L$ is an odd function.
(Hint: Rewrite $\zeta_L$ in terms of lattice points in $L$.) **[4]**

(d) Use the previous part and the fact that $\wp_L$ is elliptic to show that for each $z \in \mathbb{C}\setminus L$,

$$\zeta_L(z + \tau) = \zeta_L(z) + 2\zeta_L(\tau/2), \quad \zeta_L(z + 1) = \zeta_L(z) + 2\zeta_L(1/2).$$

(Hint: Show that $\zeta_L(z + \tau) - \zeta_L(z)$ is independent of $z$ by applying (c).) **[4]**

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**6.** Recall that the weight 2 Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

satisfies the transformation formula

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}.$$

(a) Consider the differential operator $D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that

$$(Df)(-1/\tau) = \tau^{k+2}(Df)(\tau) + \frac{1}{2\pi i}k\tau^{k+1}f(\tau).$$

(Hint: Consider $D(f(-1/\tau))$.) [**4**]

(b) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that $Df - \frac{k}{12}E_2 f \in M_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$.
Moreover, show that if $f$ is a cusp form then so is $Df - \frac{k}{12}E_2 f$. [**8**]

(c) Apply (b) to the discriminant function $\Delta(\tau) = \sum_{n=1}^{\infty}\tau(n)q^n$ to show that for every $n \in \mathbb{N}$,

$$(n-1)\tau(n) = -24\sum_{k=1}^{n-1}\sigma_1(k)\tau(n-k).$$ [**3**]

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**7.** We recall that the **Liouville function** $\lambda(n)$ is the completely multiplicative function that satisfies $\lambda(p) = -1$ for all primes $p$. In this problem you will show that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0. \tag{7.1}$$

(a) For $\mathrm{Re}(s) > 1$, define

$$D(s) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

Show that $D(s) = \zeta(2s)/\zeta(s)$ for $\mathrm{Re}(s) > 1$. **[3]**

(b) Show that $\frac{D(s)}{s-1}$ extends to a holomorphic function on the half-plane $\mathrm{Re}(s) > 1/2$. **[2]**

By the **truncated Perron formula**, it can be shown that for any $\sigma_0 > 0$ and $x, T \geq 1$,

$$\sum_{n \leq x} \frac{\lambda(n)}{n} = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} + O\left(\frac{1}{T}\left(\frac{x^{\sigma_0}}{\sigma_0} + \log T\right)\right).$$

Below, you may freely use this without proof, along with the following fact: there is a constant $c > 0$ such that whenever $s = \sigma + it$ satisfies $\sigma > 1 - \frac{c}{\log(2+|t|)}$ then

$$\frac{1}{\zeta(\sigma + it)} \ll \begin{cases} \log(2 + |t|) & \text{if } |t| \geq 1 \\ |\sigma - 1 + it| & \text{if } |t| \leq 1. \end{cases}$$

(c) Let $\alpha = \min\{1/3, c/(2\log T)\}$, where $c$ is the constant above. Show that

$$\int_{-\alpha - iT}^{-\alpha + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} = \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} + O\left(\frac{x^{\sigma_0} \log T}{T}\right).$$ **[6]**

(d) Choosing $T$ and $\sigma_0$ suitably prove that as $x \to \infty$,

$$\sum_{n \leq x} \frac{\lambda(n)}{n} = o(1).$$

Deduce that (7.1) holds. **[4]**

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**8.** Let $p$ be an odd prime and as usual, we write $(\frac{\cdot}{p})$ to denote the Legendre symbol modulo $p$. We recall that this is a non-principal Dirichlet character modulo $p$. Throughout this problem, we define the **Fekete polynomial** modulo $p$ to be the function

$$P(z) := \sum_{a=0}^{p-1} \left(\frac{a}{p}\right) z^a, \quad z \in \mathbb{C}.$$

(a) Show that for any $|z| < 1$,

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) z^n = \frac{P(z)}{1 - z^p}.$$

**[3]**

(b) Let $n \in \mathbb{N}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Prove the identity

$$\frac{1}{n^s}\Gamma(s) = \int_0^\infty t^s e^{-nt} \frac{dt}{t},$$

where $\Gamma(s)$ is the usual Gamma function.
Use this to prove the formula

$$L\left(s, \left(\tfrac{\cdot}{p}\right)\right) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) e^{-nu} \right) u^s \frac{du}{u} \tag{8.1}$$

for the Dirichlet $L$-function

$$L\left(s, \left(\frac{\cdot}{p}\right)\right) = \sum_{n \geq 1} \frac{\left(\frac{n}{p}\right)}{n^s}$$

at each $s$ with $\mathrm{Re}(s) > 1$. **[4]**

In the following problems you may assume as fact that the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) e^{-nu}$$

is uniformly bounded for all $u > 0$.

(c) Use analytic continuation to prove that the identity (8.1) extends to $\mathrm{Re}(s) > 0$. You may use the following special case of *Morera's theorem*: if a function $f$ is continuous on a half-plane $\Omega := \{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_0\}$ and satisfies

$$\int_R f(z)dz = 0$$

for every rectangle $R \subset \Omega$ then $f$ is holomorphic in $\Omega$. **[5]**

(d) Show that if $P(t) \neq 0$ for all $t \in (0, 1)$ then $L(\sigma, (\frac{\cdot}{p})) \neq 0$ for all real $\sigma \in (0, 1)$. **[3]**



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