Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH41520-WE01-SP

Title:
Topics in Algebra and Geometry V (2024/25 syllabus)

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) State the valence formula for $\mathrm{SL}_2(\mathbb{Z})$, carefully explaining all terms. **[3]**

(b) Let f and g be two non-zero meromorphic modular forms of negative weight -2 for $\mathrm{SL}_2(\mathbb{Z})$, holomorphic on the upper half plane $\mathbb{H}$ with simple poles at the cusp ∞. Show that $f = \lambda g$ for some constant λ. **[4]**

(c) Determine an explicit form of the type described in part (b). **[3]**

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2. (a) State the cocycle relation for the SL_2−automorphy factor. **[2]**

(b) Show that in order to prove that a function f on the upper half plane $\mathbb{H}$ transforms like a modular form of weight k for a group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ it is enough to do so for the generators of Γ. **[3]**

(c) Let $f(\tau)$ be a 1-periodic function on $\mathbb{H}$ such that $f(-1/\tau) = 2^{-k}\tau^k f(\tau/4)$ for some even $k \in \mathbb{Z}$.
Show that f is modular of weight k for the element $\left(\begin{smallmatrix} -1 & 0 \\ 4 & -1 \end{smallmatrix}\right) = ST^4S$. **[5]**

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3. We say that an arithmetic function $g(n)$ is **additive** if, whenever $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, we have

$$g(ab) = g(a) + g(b).$$

(a) Recall that $\omega(n) := |\{p \text{ prime} : \ p|n\}|$. Show that $\omega(n)$ is an additive function. **[3]**

(b) Show that if $f(n)$ is a multiplicative function taking positive real values then $\log f(n)$ is an additive function. **[3]**

(c) Suppose that $g : \mathbb{N} \to \mathbb{C}$ is an additive function. Let $n > 1$ be an integer with prime factorisation of the form $n = \prod_{i=1}^{r} p_i^{k_i}$, where p_i's are distinct primes. Show that

$$g(n) = \sum_{i=1}^{r} g(p_i^{k_i}).$$ **[4]**

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4. Recall that a positive integer n is called **squarefree** if, for any prime p that divides n, we have $p^2 \nmid n$.

(a) Show that

$$|\{n \leq x : \ n \text{ is squarefree}\}| = \sum_{n \leq x} \mu^2(n),$$

where $\mu(n)$ denotes the Möbius function. **[2]**

(b) Using the identity

$$\mu^2(n) = \sum_{d^2 | n} \mu(d), \quad n \geq 1,$$

prove the estimate

$$|\{n \leq x : \ n \text{ is squarefree}\}| = x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} + O(\sqrt{x}).$$ **[4]**

(c) Recall that for $\mathrm{Re}(s) > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Use this to prove that

$$\sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\frac{1}{\sqrt{x}}\right)$$

and hence deduce that

$$\lim_{x \to \infty} \frac{1}{x} |\{n \leq x : \ n \text{ is squarefree}\}| = \frac{1}{\zeta(2)}.$$ **[4]**

SECTION B

5. The Leech lattice L is an even positive definite unimodular lattice of rank 24 which contains no element of length 1. That is, for the underlying quadratic form Q we have

$$Q(\mathbf{x}) \neq 1 \qquad \text{for all } \mathbf{x} \in L. \tag{5.1}$$

(a) Carefully define the associated theta series $\theta(\tau, L)$ and explain condition (5.1) in terms of $\theta(\tau, L)$. **[3]**

(b) Express $\theta(\tau, L)$ in terms of the Eisenstein series

$$E_{12}(\tau) = 1 + \tfrac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

and the discriminant function

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Carefully state what results you use. **[8]**

(c) Use (b) to derive the Ramanujan congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$ **[4]**

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6. Recall that the weight 2 Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

satisfies the transformation formula

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}.$$

(a) Consider the differential operator $D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that

$$(Df)(-1/\tau) = \tau^{k+2}(Df)(\tau) + \frac{1}{2\pi i} k \tau^{k+1} f(\tau).$$

(Hint: Consider $D(f(-1/\tau))$.) **[4]**

(b) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that $Df - \frac{k}{12} E_2 f \in M_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$. Moreover, show that if f is a cusp form then so is $Df - \frac{k}{12} E_2 f$. **[8]**

(c) Apply (b) to the discriminant function $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$ to show that for every $n \in \mathbb{N}$,

$$(n-1)\tau(n) = -24 \sum_{k=1}^{n} \sigma_1(k) \tau(n-k).$$ **[3]**

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7. We recall that the **Liouville function** $\lambda(n)$ is the completely multiplicative function that satisfies $\lambda(p) = -1$ for all primes p. In this problem you will show that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0. \tag{7.1}$$

(a) For $\mathrm{Re}(s) > 1$, define

$$D(s) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

Show that $D(s) = \zeta(2s)/\zeta(s)$ for $\mathrm{Re}(s) > 1$. **[3]**

(b) Show that $\frac{D(s)}{s-1}$ extends to a holomorphic function on the half-plane $\mathrm{Re}(s) > 1/2$. **[2]**

By the **truncated Perron formula**, it can be shown that for any $\sigma_0 > 0$ and $x, T \geq 1$,

$$\sum_{n \leq x} \frac{\lambda(n)}{n} = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} + O\left(\frac{1}{T}\left(\frac{x^{\sigma_0}}{\sigma_0} + \log T\right)\right).$$

Below, you may freely use the above formula and the following fact without proof: there is a constant $c > 0$ such that whenever $s = \sigma + it$ satisfies $\sigma > 1 - \frac{c}{\log(2+|t|)}$ then

$$\frac{1}{\zeta(\sigma + it)} \ll \begin{cases} \log(2 + |t|) & \text{if } |t| \geq 1 \\ |\sigma - 1 + it| & \text{if } |t| \leq 1. \end{cases}$$

(c) Let $\alpha = \min\{1/3, c/(2\log T)\}$, where c is the constant above. Show that

$$\int_{-\alpha - iT}^{-\alpha + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} = \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(2s+2)}{\zeta(s+1)} x^s \frac{ds}{s} + O\left(\frac{x^{\sigma_0} \log T}{T}\right).$$ **[6]**

(d) Choosing T and σ_0 suitably prove that as $x \to \infty$,

$$\sum_{n \leq x} \frac{\lambda(n)}{n} = o(1).$$

Deduce that (7.1) holds. **[4]**

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8. Let q be a positive integer and given $x \in \mathbb{R}$, let $e(x) := \exp(2\pi i x)$.

(a) Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a Dirichlet character modulo q. **[2]**

(b) Using the formula for a geometric sum or otherwise, prove that for any integer k,

$$\sum_{m=1}^{q} e\left(\frac{mk}{q}\right) = \begin{cases} q & \text{if } k \equiv 0 \pmod q \\ 0 & \text{otherwise .} \end{cases}$$ **[3]**

(c) If χ is a Dirichlet character modulo q, then define the function $\hat{\chi} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{\chi}(m) = \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) e\left(-\frac{mn}{q}\right).$$

Prove that if χ is a Dirichlet character modulo q then

$$\chi(n) = \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \hat{\chi}(m) e\left(\frac{mn}{q}\right).$$ **[4]**

(d) Let χ be a Dirichlet character modulo q and let $\gcd(m, q) = 1$. Prove that

$$\hat{\chi}(m) = \overline{\chi}(m)\hat{\chi}(1).$$

Further prove that $\hat{\chi}$ is not necessarily a Dirichlet character modulo q. **[6]**
