



## EXAMINATION PAPER

<b>Examination Session:</b> May/June	<b>Year:</b> 2026	<b>Exam Code:</b> MATH4161-WE01
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<b>Title:</b> Algebraic Topology IV
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Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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<b>Revision:</b>	
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## SECTION A

1. This question is about short exact sequences of abelian groups.

- (a) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that  $A \cong D$  and  $B \cong E$ , but  $C \not\cong F$ .

[3]

- (b) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that  $A \cong D$  and  $C \cong F$ , but  $B \not\cong E$ .

[2]

- (c) Give an example of two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

such that  $B \cong E$  and  $C \cong F$ , but  $A \not\cong D$ .

[5]

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2. Let  $S^2 \subset \mathbb{R}^3$  be the unit 2-sphere inside 3-dimensional space. Compute the homology groups of the quotient space  $H_n(\mathbb{R}^3/S^2)$  for all  $n \geq 0$ .

[10]

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3. For  $n \in \mathbb{Z}$ , define  $C_n = \mathbb{Z}/18\mathbb{Z}$  and let  $\partial_n: C_n \rightarrow C_{n-1}$  be given by  $\partial_n(\bar{k}) = \overline{6k}$ .

- (a) Show that  $(C_*, \partial)$  is a chain complex and calculate  $H_n(C_*)$ .

[3]

- (b) Calculate  $H^n(\text{Hom}(C_*, \mathbb{Z}/4\mathbb{Z}))$  for all  $n \in \mathbb{Z}$ .

[3]

- (c) Calculate  $H^n(\text{Hom}(C_*, \mathbb{Z}/9\mathbb{Z}))$  for all  $n \in \mathbb{Z}$ .

[4]

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4. Let  $M$  be a compact, connected manifold of dimension  $m$ , possibly with boundary.

- (a) State the definition of  $M$  being  $\mathbb{Z}$ -orientable.

[3]

- (b) Assume that  $\partial M \neq \emptyset$ . Show that  $H_m(M) = 0$ . You may assume that  $M$  admits a triangulation. State any results from the lectures that you use.

[7]

## SECTION B

5. (a) Using a Mayer-Vietoris sequence or otherwise, compute the homology groups of the space  $Y$  given as a subspace of  $\mathbb{R}^3$  as the union of the unit 2-sphere together with the unit coordinate axes:

$$\begin{aligned} Y = & \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \\ & \cup \{(x, 0, 0) \in \mathbb{R}^3 : -1 \leq x \leq 1\} \\ & \cup \{(0, y, 0) \in \mathbb{R}^3 : -1 \leq y \leq 1\} \\ & \cup \{(0, 0, z) \in \mathbb{R}^3 : -1 \leq z \leq 1\}. \end{aligned} \quad [6]$$

- (b) Assuming that the homology groups  $H_n(X)$  are free abelian for each  $n \in \mathbb{Z}$ , show using a Mayer-Vietoris long exact sequence that

$$H_n(S^1 \times X) \cong H_{n-1}(X) \oplus H_n(X)$$

for all  $n \in \mathbb{Z}$ . [9]

6. Suppose that  $(A_*, d_*^A)$ ,  $(B_*, d_*^B)$ , and  $(C_*, d_*^C)$  are chain complexes of free abelian groups. Suppose further that

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

is a short exact sequence of these chain complexes (where  $f_*$  and  $g_*$  are chain maps).

- (a) In such a situation there is a *connecting homomorphism*

$$\delta_n : H_n(C_*) \longrightarrow H_{n-1}(A_*)$$

which fits into a long exact sequence of homology groups by the snake lemma.

If  $[c] \in \ker(d_n^C)/\text{im}(d_{n-1}^C) = H_n(C_*)$ , explain how to construct the element  $\delta_n([c]) \in H_{n-1}(A_*)$  (you need not show that this element is well-defined, and you need not derive any of the properties of the long exact sequence). [5]

- (b) Give an example of a short exact sequence of chain complexes of free abelian groups

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

in which  $H_1(C_*) = H_0(A_*) = \mathbb{Z}/4\mathbb{Z}$  and for which the connecting homomorphism satisfies  $\delta_1(m) = 2m$  for all  $m \in \mathbb{Z}/4\mathbb{Z}$ . [10]

7. Recall that the Universal Coefficient Theorem gives a short exact sequence

$$0 \longrightarrow \text{Ext}(H_{n-1}(X), \mathbb{K}) \longrightarrow H^n(X; \mathbb{K}) \xrightarrow{\Phi} \text{Hom}(H_n(X), \mathbb{K}) \longrightarrow 0$$

where  $X$  is a topological space,  $\mathbb{K}$  a commutative ring, and  $\Phi$  the map induced by the Kronecker product.

(a) Explain what it means that this short exact sequence is *natural*. [4]

(b) The Universal Coefficient Theorem also states that the above short exact sequence splits. Explain what this means. [2]

(c) Consider the quotient map  $f: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2/\mathbb{R}P^1$ . Show that

$$f^*: H^2(\mathbb{R}P^2/\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism. Here  $\mathbb{R}P^n$  is projective  $n$ -space. [4]

(d) By using part (c), or otherwise, provide an example to show that the splitting of the above sequence is not natural in general. [5]

8. Let  $S^n \subset \mathbb{R}^{n+1}$  be the unit  $n$ -sphere inside  $(n + 1)$ -dimensional space.

(a) Describe the cohomology ring structure of  $S^2 \times S^2$  without proof. [4]

(b) Give an example of a map  $f: S^2 \times S^2 \rightarrow S^4$  which induces an isomorphism

$$f^*: H^4(S^4; \mathbb{Z}) \rightarrow H^4(S^2 \times S^2; \mathbb{Z}),$$

and justify your statement. [6]

(c) Let  $g: S^4 \rightarrow S^2 \times S^2$  be any map. Show that

$$g^*: H^4(S^2 \times S^2; \mathbb{Z}) \rightarrow H^4(S^4; \mathbb{Z})$$

is the 0-map. [5]