



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH41620-WE01
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Title: Number Theory V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. (a) Show that there is no solution in integers other than $x = y = z = 0$ to

$$3x^{11} + 5y^{11} = 7z^{11}.$$

[5]

- (b) For $m \in \mathbb{Z}$ show that

$$\frac{m - \sqrt[3]{m}}{\sqrt[3]{3}}$$

is an algebraic integer.

[5]

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2. Let $K = \mathbb{Q}(\sqrt{-23})$ and let $R = \mathcal{O}_{-23}$ be its ring of integers.

- (a) Show that the ideal $J = (1 + 5\sqrt{-23}, 1 + \sqrt{-23})_R \subset R$ is not a principal ideal.

[4]

- (b) Compute the ideal J^2 and write it with at most two generators, where one of the generators is an integer.

[2]

- (c) Put $I = (\frac{1+\sqrt{-23}}{2}, 3)_R$. Show that $R/I \simeq \mathbb{Z}/3$. (Carefully state any theorem that you use.)

[4]

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3. Let $\theta \in \mathbb{C}$ be a root of $f(x) = x^3 + 2x + 2$ and let $K = \mathbb{Q}(\theta)$. In this question, you may assume that $R = \mathbb{Z}[\theta]$ is the ring of integers of K .

- (a) Find $[K : \mathbb{Q}]$ and compute $N_K(4 - \theta)$.

[3]

- (b) Find two distinct prime ideals of R (given in terms of generators) that divide $(N_K(4 - \theta))_R$.

[4]

- (c) Factorise the ideal $(4 - \theta)_R$ as a product of prime ideals.

[3]

SECTION B

4. (a) Give the ring of integers \mathcal{O}_{61} in $\mathbb{Q}(\sqrt{61})$ and find its fundamental unit. [5]

- (b) Find a solution (a, b) with positive integers a, b such that (note the positive sign on the right)

$$a^2 - 61b^2 = 4.$$

[3]

- (d) Give all the solutions in integers x, y , if any, to the equation

$$x^2 - 18y^2 = 7.$$

[7]

5. Let $R = \mathbb{Z}[\sqrt{-2}]$.

- (a) Show that R is a Euclidean domain. [5]

- (b) Write 11 and 59 in $\mathbb{Z}[\sqrt{-2}]$ as a product of irreducible elements in all possible ways, up to associates. [2]

- (c) Determine the number of different $\alpha \in R$ dividing 649. [3]

- (d) Using the previous parts, or otherwise, find all the solutions to the equation [5]

$$a^2 + 2b^2 = 649$$

in integers.

6. Let $\theta \in \mathbb{C}$ be such that $\theta^3 = 9$ and let $K = \mathbb{Q}(\theta)$.

- (a) Find the discriminant $\Delta_K(1, \theta, \theta^2)$. [3]

- (b) Show that $\mathbb{Z}[\theta^2/3]$ is a full lattice in K by finding a generating basis. Find the discriminant $\Delta_K(\mathbb{Z}[\theta^2/3])$. [4]

- (c) Show that $\theta^2/3 \in \mathcal{O}_K$ and that $\mathcal{O}_K = \mathbb{Z}[\theta^2/3]$. [8]

7. Let $\theta = \sqrt{-30}$, $K = \mathbb{Q}(\theta)$ and $R = \mathcal{O}_K$.

(a) Determine the class group $Cl(R)$. You may use the Minkowski bound, given

$$\text{by } B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n} \sqrt{|\Delta_K|}. \quad [6]$$

(b) Let \mathfrak{p}_{11} be any prime ideal of R above 11. In your computation of $Cl(R)$ in the previous part, find the ideal class in $Cl(R)$ that \mathfrak{p}_{11} belongs to. In other words, find the element of $Cl(R)$ that is equal to $[\mathfrak{p}_{11}]$. [9]

SECTION C

8. (a) Write $\frac{1}{16}$ as a 5-adic expansion, that is, in the form $a_0 a_1 \dots \overline{a_r a_{r+1} \dots a_{r+s}}$ for some $r, s \geq 1$, where $\frac{1}{16} = \sum_{i=0}^{\infty} a_i 5^i$, for $a_i \in \{0, 1, 2, 3, 4\}$. Give a_0, \dots, a_5 explicitly. [3]

(b) Show that the geometric series $\sum_{n=0}^{\infty} p^n$ converges in \mathbb{Q}_p and determine its limit. [2]

(c) For an odd prime p find the p -adic expansion of

$$\frac{1}{(1-p)^2}.$$

Use this expansion to recover the 5-adic expansion from part (a). [5]