Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4171-WE01

Title:
Riemannian Geometry IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Prove or disprove the following assertions:

(a) *The subset of* $\mathbb{R}^2$ *given by*

$$M = \{\, (x, y) \in \mathbb{R}^2 : |y| = |x| \,\}$$

is a smooth embedded submanifold of $\mathbb{R}^2$. **[5]**

(b) *The subset of* $\mathbb{R}^4$ *given by*

$$M = \{(w, x, y, z) \in \mathbb{R}^4 : w + e^x + 3y + 2z^4 = 0\}$$

is a smooth 3*-dimensional submanifold of* $\mathbb{R}^4$. **[5]**

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2. (a) Prove or disprove the following assertion: *there exists a Riemannian metric g on the real projective plane $\mathbb{R}P^2$ such that, for some chart $(U, (x^1, x^2))$, the local expression of g on U is given by*

$$(g_{ij}) = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}.$$

[4]

(b) State the definition of a diffeomorphism between smooth manifolds. **[2]**

(c) Let M and N be 3-smooth manifolds and let $f\colon M \to N$ be a smooth function whose differential in local coordinates around $p \in M$ and $f(p) \in N$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Can f be a diffeomorphism? Justify your answer. **[4]**

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3. Let ∇^1 and ∇^2 be affine connections on a smooth manifold M.

(a) Prove or disprove the following assertion: *the map*

$$\begin{aligned}\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) &\to \mathcal{X}(M)\\ (X,Y) &\mapsto \nabla^1_X Y + \nabla^2_X Y\end{aligned}$$

defines an affine connection on M, where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M. **[5]**

(b) Let T^1 and T^2 denote the torsion tensors of ∇^1 and ∇^2, respectively, and suppose that ∇^1 and ∇^2 are compatible with a Riemannian metric g on M. The difference tensor for ∇^1 and ∇^2 is given by

$$\begin{aligned}A\colon \mathcal{X}(M) \times \mathcal{X}(M) &\to \mathcal{X}(M)\\ (X,Y) &\mapsto \nabla^1_X Y - \nabla^2_X Y.\end{aligned}$$

Show that the difference tensor A satisfies

(i) $A(X,Y) - A(Y,X) = T^1(X,Y) - T^2(X,Y)$, and

(ii) $g(A(X,Y),Z) = -g(Y,A(X,Z))$

for all $X,Y,Z \in \mathcal{X}(M)$. **[5]**

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4. Prove or disprove the following assertions. You may use, without proof, any relevant results from the lectures.

(a) *If (M,g) is a complete, path-connected Riemannian manifold with Riemannian distance d_g and $p,q \in M$ are points in M, then there exists a piecewise-smooth curve $c\colon [a,b] \to M$ with $c(a) = p$, $c(b) = q$ and length $L(c) = d_g(p,q)$.* **[2]**

(b) *The n-dimensional sphere S^n admits a complete Riemannian metric with sectional curvature $K \leq 0$.* **[4]**

(c) *There exist positive integers n and k such that the smooth product manifold $S^n \times \mathbb{R}^k$ admits a complete Riemannian metric with sectional curvature $K \geq 3$.* **[4]**

SECTION B

5. Let

$$X(x,y,z) = z\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y} + (2y-x)\frac{\partial}{\partial z},$$
$$Y(x,y,z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

be two smooth vector fields on $\mathbb{R}^3$ equipped with the usual chart given by the identity map.

(a) Compute the Lie bracket $[X, Y]$. **[5]**

(b) Let $S^2(1) \subset \mathbb{R}^3$ be the unit round sphere centred at the origin. Show that the restrictions of the vector fields X and Y to $S^2(1)$ are vector fields on $S^2(1)$, i.e. $X(p)$ and $Y(p)$ are tangent to $S^2(1)$ for any $p \in S^2(1)$. **[6]**

(c) Show that the restriction of $[X, Y]$ to $S^2(1)$ is also a vector field on $S^2(1)$. **[4]**

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6. (a) State the definition of a Lie group. **[4]**

(b) Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices. Show that the general linear group $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ is a Lie group. **[6]**

(c) Let (M, g) be a Riemannian manifold and let X be a smooth vector field on M. Show that $p \mapsto g_p(X(p), \cdot\,)$ is a smooth differential form on M. **[5]**

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7. Let (M_1, g_1) and (M_2, g_2) be path-connected Riemannian manifolds and suppose that there exists an isometry $f\colon (M_1, g_1) \to (M_2, g_2)$. Let $c\colon [a,b] \to M_1$ be a piecewise smooth curve.

(a) Show that
$$L(c) = L(f \circ c),$$
where $L(c)$ and $L(f \circ c)$ are the lengths of the curves c and $f \circ c$ in (M_1, g_1) and (M_2, g_2) respectively. **[2]**

(b) Show that
$$d_{g_1}(c(a), c(b)) = d_{g_2}(f(c(a)), f(c(b))),$$
where d_{g_1} and d_{g_2} denote the Riemannian distances in (M_1, g_1) and (M_2, g_2) respectively. **[4]**

(c) If $c\colon [a,b] \to M_1$ is a minimising geodesic segment, show that the curve $f \circ c\colon [a,b] \to M_2$ is a minimising geodesic segment and is given by
$$f(c(t)) = \exp_{f(c(a))}\left(t\, Df_{f(c(a))}(c'(a))\right), \quad t \in [a,b].$$ **[4]**

(d) Let $I \in \mathbb{R}$ be an open interval and let $\gamma\colon I \to M_1$ be a geodesic in (M_1, g_1). Show that $f \circ \gamma\colon I \to M_2$ is a geodesic in (M_2, g_2). **[2]**

(e) Show that (M_1, g_1) is geodesically complete if and only if (M_2, g_2) is geodesically complete. **[3]**

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8. Let (M, g) be a three-dimensional Riemannian manifold with Levi-Civita connection ∇. Suppose that there exist smooth vector fields $X_1, X_2, X_3 \in \mathcal{X}(M)$ on M such that

$$g(X_i, X_j) = \delta_{ij},$$

for all $i, j \in \{1, 2, 3\}$, where $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$. Suppose, furthermore, that

$$\nabla_{X_1} X_2 = X_3, \quad \nabla_{X_2} X_3 = X_1, \quad \nabla_{X_3} X_1 = X_2 \quad \text{and} \quad \nabla_{X_i} X_j = -\nabla_{X_j} X_i,$$

for all $i, j \in \{1, 2, 3\}$.

(a) Show that the Lie bracket satisfies

$$[X_i, X_j] = \begin{cases} 2X_k, & \text{if } (i,j,k) \in \{(1,2,3), (2,3,1), (3,1,2)\}, \\ -2X_k, & \text{if } (i,j,k) \in \{(2,1,3), (1,3,2), (3,2,1)\}. \end{cases}$$

Note that this implies that the vector fields X_1, X_2, X_3 are not coordinate vector fields. **[3]**

(b) Show that the curvature tensor R of (M, g) satisfies

$$R(X_i, X_j)X_k = \begin{cases} X_i, & \text{if } i, j, k \in \{1, 2, 3\} \text{ with } i \neq j,\ j = k, \\ -X_j, & \text{if } i, j, k \in \{1, 2, 3\} \text{ with } i \neq j,\ i = k, \\ 0, & \text{otherwise.} \end{cases}$$

[5]

(c) Show that the sectional curvature K satisfies

$$K(X_i, X_j) = 1,$$

for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. **[2]**

(d) Show that (M, g) has constant sectional curvature; that is, show that

$$K(v, w) = 1,$$

for all $p \in M$ and all $v, w \in T_pM$ with $\|v\| = 1$, $\|w\| = 1$ and $g_p(v, w) = 0$. **[5]**
