EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH41720-WE01

Title:
Partial Differential Equations V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Consider the PDE

$$\begin{cases} x\partial_x u(x,y) + y\partial_y u(x,y) = 0, & (x,y) \in \mathbb{R}^2, \\ u(x,5) = x, & x \in \mathbb{R}. \end{cases}$$

(a) Find the Cauchy curve Γ associated to the problem and all the points on it which are non-characteristics. **[3]**

(b) Find a solution to the equation using the method of characteristics and determine its maximal domain of definition. **[7]**

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2. Consider the system

$$u_t(x,t) = u_{xx}(x,t) - u(x,t) \quad \text{and} \quad u(x,0) = u_0(x)$$

for $x \in \mathbb{R}$ and $t \geq 0$.

(a) Write down a differential equation for the Fourier transform $\hat{u}(\xi, t)$. **[2]**

(b) Solve the equation for $\hat{u}(\xi, t)$, and use it to obtain an (explicit) integral representation for $u(x,t)$ in terms of $u_0(x)$. **[4]**

(c) Given that $u_0 \in L^2(\mathbb{R})$, show that $u(\cdot, t) \in L^2(\mathbb{R})$ for all $t \geq 0$. Cite any results you use (no need to prove them). **[4]**

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3. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with smooth boundary, and let m, $f : \Omega \to \mathbb{R}$ be given with $m(x) > 0$. For $u \in V := \{v \in C^1(\bar{\Omega}) : v = 0 \text{ on } \partial\Omega\}$, we put

$$E[u] := \int_\Omega \big\{ |\nabla u(x)|^2 + m(x)u(x)^2 - 2f(x)u(x) \big\} \, \mathrm{d}x.$$

(a) Assuming that $u \in V$ is a minimiser of $E[u]$, write down the equation that it must satisfy (showing that u is some weak solution). **[4]**

(b) Assuming further that $u \in V \cap C^2(\bar{\Omega})$, write down the partial differential equation and boundary conditions satisfied by u. **[4]**

(c) Assuming a minimiser exists, is it unique? Justify your answer. **[2]**

SECTION B

4. Consider Burgers' equation

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$u_0(x) = \begin{cases} 0, & x < 0, \\ x+1, & x > 0. \end{cases}$$

(a) Find and draw the characteristics associated to the problem. **[3]**

(b) Show that

$$u(x,t) = \begin{cases} 0, & x < \sqrt{1+t} - 1, \quad t > 0, \\ \frac{1+x}{1+t}, & x > \sqrt{1+t} - 1, \quad t > 0. \end{cases}$$

is a weak integral solution for our problem. Justify your answer. **[10]**

(c) Does the above solution satisfy Lax's entropy condition? Justify your answer. **[2]**

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5. In this question we will see that we can utilise the method of characteristics to solve a linear system of first order differential equations when the leading vector field is the same throughout the system.

Consider the system of equations

$$\begin{cases} 3\partial_x u_1(x,y) + 6x\partial_y u_1(x,y) = u_1(x,y), & (x,y) \in \mathbb{R}^2. \\ 3\partial_x u_2(x,y) + 6x\partial_y u_2(x,y) = u_1^2(x,y), & (x,y) \in \mathbb{R}^2, \\ u_1(0,y) = 1, & y \in \mathbb{R}, \\ u_2(0,y) = y, & y \in \mathbb{R}. \end{cases}$$

(a) Check that every point on the Cauchy curve $\Gamma = \{(0,y) \mid y \in \mathbb{R}\}$ is non-characteristics with respect to the leading vector field

$$\vec{a}\,(x,y) = (3, 6x)\,.$$

[2]

(b) Find the characteristics $(X(\tau,s), Y(\tau,s))$ which are associated to the given leading vector field. **[4]**

(c) Show that

$$z_1(\tau,s) = u_1(X(\tau,s), Y(\tau,s)) \quad \text{and} \quad z_2(\tau,s) = u_2(X(\tau,s), Y(\tau,s)),$$

where $(X(\tau,s), Y(\tau,s))$ are the characteristics found in part (**5.**b), are given by **[5]**

$$z_1(\tau,s) = e^{\tau}, \quad z_2(\tau,s) = s + \frac{e^{2\tau}-1}{2}.$$

(d) Find a solution to the system of PDEs, $u_1(x,y)$ and $u_2(x,y)$. Where is it defined? **[4]**

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6. Let $\Omega \subset \mathbb{R}^n$ be connected and bounded with smooth boundary.

(a) Prove that if u is harmonic in Ω, then

$$\int_{\partial\Omega} \nabla u \cdot \, \mathrm{d}S = 0.$$

[2]

(b) Use the above fact to prove the mean-value formulas:

$$\frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y) = u(x) \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,\mathrm{d}y = u(x)$$

where $|B_r|$ and $|\partial B_r|$ are the "volume" of a ball and a sphere of radius r. **[4]**

(c) State and prove the *weak maximum principle* for harmonic functions. **[5]**

(d) Using the weak maximum principle, prove that a solution u to the problem

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ in } \partial\Omega$$

is unique if it exists. You may assume that $f \in C^2(\bar{\Omega})$ and $g \in C^1(\partial\Omega)$. **[4]**

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7. Consider the problem

$$u_t(x,t) = -u_{xxxx}(x,t) + f(x),$$
$$u(a,t) = u(b,t) = u_x(a,t) = u_x(b,t) = 0$$

where $x \in [a,b]$ and $t \geq 0$, and $f \in C([a,b])$ and $u(x,0) = u_0(x)$ are given.

(a) For any $w \in C^2([a,b])$ with $w(a) = w(b) = w'(a) = w'(b) = 0$, prove that

$$\|w\|_{L^2} \leq c_p\|w'\|_{L^2} \leq c_p^2\|w''\|_{L^2}$$

for some (positive) constant c_p. **[3]**

(b) Assuming that u is a classical solution to the above problem, compute a bound for $\|u(\cdot,t)\|_{L^2}$ of the form

$$\|u(\cdot,t)\|_{L^2}^2 \leq \mathrm{e}^{-c_1 t}\|u_0\|_{L^2}^2 + c_2\,\|f\|_{L^2}^2$$

for some constants c_1, $c_2 > 0$.Hint: multiply by u and integrate by parts. **[6]**

(c) Let $v(x)$ be the classical solution of

$$v_{xxxx}(x) = f(x)$$

with $v(a) = v(b) = v_x(a) = v_x(b) = 0$. Show that $u(\cdot,t) \to v$ in $L^2([a,b])$ as $t \to \infty$. **[6]**

SECTION C

8. Consider the sequence of regular distributions $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R})$ whose L^1_{loc} representatives are given by

$$\widetilde{f_n}(x) = \begin{cases} e^{-nx}, & 0 < x < n, \\ 0, & x < 0 \text{ or } x > n, \end{cases}$$

i.e. the distributions $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R})$ are such that

$$(f_n, \phi) = \int_{\mathbb{R}} \widetilde{f_n}(x)\phi(x)dx,$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

(a) Show that $\{f_n\}_{n\in\mathbb{N}}$ converges in the sense of distribution to $f = 0$. **[5]**

(b) Use the definition of distributional derivatives to show that

$$f_n' = g_n + \delta_0 - e^{-n^2}\delta_n$$

where g_n is a regular distribution with representative

$$\widetilde{g_n}(x) = \begin{cases} -ne^{-nx}, & 0 < x < n, \\ 0, & x < 0,\ x > n, \end{cases}$$

and δ_a is the delta distribution that places a unit mass at the point a. **[5]**
