Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH41920-WE01

Title: Geometry V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Is it true or false that:

(a) The group of isometries of the Euclidean plane can be generated by all glide reflections? Justify your answer. **[5]**

(b) The group of orientation-preserving isometries of the Euclidean plane can be generated by all translations? Justify your answer. **[5]**

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2. Let $A_1A_2A_3$ be a triangle in the Euclidean plane. Denote also $A_4 = A_1$. Let B_i, C_i, D_i, $i = 1, 2, 3$ be points on A_iA_{i+1} such that $A_iB_i = B_iC_i = C_iD_i = D_iA_{i+1}$.

(a) Let B_0 be the common point of the medians of $\triangle B_1B_2B_3$ and let C_0 be the common point of the medians of $\triangle C_1C_2C_3$. Show that $B_0 = C_0$. **[5]**

(b) Let $\widehat{B}$ be the common point of the altitudes of $\triangle B_1B_2B_3$ and let $\widehat{C}$ be the common point of the altitudes of $\triangle C_1C_2C_3$. Is it always true that $\widehat{B} = \widehat{C}$? **[5]**

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3. Let $A_1A_2A_3A_4A_5A_6$ be an ideal hexagon in the hyperbolic plane (i.e. a hexagon with all vertices at the boundary). Denote also $A_7 = A_1$. Assume that there exists a point O inside this hexagon such that $\angle A_iOA_{i+1} = \frac{\pi}{3}$ for $i = 1, \dots, 6$.

(a) Show that there exists a circle inscribed in $A_1A_2A_3A_4A_5A_6$. **[5]**

(b) Let r be the radius of the circle inscribed in $A_1A_2A_3A_4A_5A_6$. Find $\cosh r$. **[5]**

SECTION B

4. Let γ_0 be a circle in the Euclidean plane and let A, B, C, D be distinct points of γ_0 listed in cyclic order. Let γ_1 be the circle through A and C orthogonal to γ_0. Let γ_2 be the circle through B and D orthogonal to γ_0. Assume $[A, B, C, D] = 2$.

(a) Show that the circles γ_1 and γ_2 intersect and find the angle between them. **[5]**

(b) Let i_0, i_1, i_2 be inversions with respect to $\gamma_0, \gamma_1, \gamma_2$ respectively. Let G be the group generated by i_0i_1 and i_0i_2. Show that G is a finite subgroup of the group of Möbius transformations on $\mathbb{C} \cup \{\infty\}$. **[5]**

(c) Show that all the non-trivial elements of the group G defined in part (b) are elliptic. Find all possible orders of $g \in G$. **[5]**

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5. (a) Let A and B be points in the Euclidean plane, let a_1, a_2 be lines through A and $b_1, \dots, b_4$ be lines through B. Denote $S_{i,j} = a_i \cap b_j$, and consider the lines $p_i = S_{1,i}S_{2,i+1}$ for $i = 1, 2, 3$ and $q_i = S_{2,i}S_{1,i+1}$ for $i = 1, 2, 3$. Assume that none of the lines p_i and q_j are parallel to each other. Show that the lines p_1, p_2, p_3 are concurrent if and only if the lines q_1, q_2, q_3 are concurrent. **[5]**

(b) Formulate the statement dual to the one stated in part (a). **[5]**

(c) Is it true that the group of projective transformations acts transitively on the configurations described in part (a) and satisfying the condition that the lines p_1, p_2, p_3 are concurrent? Justify your answer. **[5]**

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6. Let h_1 and h_2 be two distinct horocycles centred at the same point of the boundary of the hyperbolic plane.

(a) Show that there exists a set of hyperbolic circles C_i, $i \in \mathbb{Z}$ such that all C_i are tangent to both h_1 and h_2 and C_i is tangent to C_{i+1} for all $i \in \mathbb{Z}$. **[5]**

(b) Let O_i be the (hyperbolic) centre of the circle C_i described in part (a), let $A_i = C_{i-1} \cap C_i$. Which of the (hyperbolic) segments is shorter, O_iO_{i+1} or A_iA_{i+1}? Justify your answer. **[5]**

(c) In the above notation, assume that $\angle A_iO_iA_{i+1} = \pi/2$. Find the (hyperbolic) radius of C_i. **[5]**

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7. (a) Let H_1 and H_2 be two hyperbolic hexagons. Assume that each of H_1 and H_2 has six right angles. Is it always true that there exists an isometry of the hyperbolic plane taking H_1 to H_2? Justify your answer. **[5]**

(b) Suppose that $ABCDEF$ is a hyperbolic hexagon with six right angles, suppose also that the opposite sides of $ABCDEF$ are equal. Show that the diagonals AD, BE and CF have a common point O. **[5]**

(c) Let $ABCDEF$ be a right-angled hyperbolic hexagon with equal opposite sides and O as in part (b). Is it true or false that the area of $\triangle OAB$ is always equal to the area of $\triangle OBC$? Justify your answer. **[5]**

SECTION C

8. (a) A curve C of the second order is given by $x_1^2 - 2x_1 + x_2^2 = 0$. Give an equation of a curve C' of the second order which is the same as C in the projective classification but different in the affine classification. Give an equation of a curve C'' of the second order which is different from C even in the projective classification. Justify your answer. **[5]**

(b) An ellipse is drawn in the plane (neither foci nor the centre are marked). Using ruler and compass, find the centre of the ellipse. Describe the construction and explain why it works.

(You do not need to use any particular instruments to draw your diagrams. You can use without explanations and further details the following elementary constructions: midpoint of a segment, perpendicular bisector for a segment.) **[5]**
