Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4231-WE01

Title:
Statistical Mechanics IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. A magnet of magnetisation M in an external magnetic field B has an internal energy and an equation of state given by

$$E = \alpha T \, , \qquad T = \frac{\gamma B}{M} \, .$$

Here α and γ are constants. You can assume that all of the processes in this question are reversible, and the first law of thermodynamics takes the form $dE = TdS + đW$, where $đW = BdM$.

(a) For this system, compute the change in energy, the work performed, and the change in entropy for an isothermal process at constant temperature T_0 between (M_0, B_0) and (M_1, B_1). **[4]**

(b) Find a differential equation of the form $\frac{dB}{dM} = f(B, M)$ satisfied by an adiabat in the (B, M) plane. (You do not need to solve this differential equation, but you should work out the form of $f(B, M)$). **[4]**

(c) A door-to-door salesperson knocks on your door, claiming to have invented a new type of engine using this magnet that operates between a hot reservoir at temperature of $320K$ and a cold reservoir at room temperature $(300K)$ and is 90% efficient (i.e., has an efficiency of $\eta = 0.9$). Should you trust the salesperson? Why or why not? **[2]**

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2. Consider the Hamiltonian of a free particle on the torus T^2:

$$H(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} \, ,$$

where $q_1, q_2 \in [0, 2\pi L]$ with periodic identification, while p_1 and p_2 can be any real numbers. The phase space density in the microcanonical ensemble is given by

$$\rho(E, L) d\mu = \frac{1}{\mathcal{N}} \delta(H - E) d\mu \, ,$$

where $\mathcal{N} = \mathcal{N}(E, L)$ ensures that ρ is a properly normalised probability distribution on phase space, $d\mu = \prod_i dq_i dp_i$, and you may assume $E > 0$.

(a) Compute in the microcanonical ensemble the entropy $S(E) = k_B \log \Omega(E)$, where $\Omega(E) = \frac{\mathcal{N}}{\mathcal{N}_0}$, and we take $\mathcal{N}_0 = \frac{h^2}{E}$, with h^2 a constant. Compute the temperature T as a function of E. **[6]**

(b) Determine the unconditional probability density function $\rho_{\text{unc}}(p_1)$ in the microcanonical ensemble. You do not need to normalise it correctly: in this problem we are only asking for the p_1 dependence. You may assume $p_1^2 < 2mE$. **[3]**

(c) What is $\langle p_1^2 \rangle$ in the microcanonical ensemble? **[1]**

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3. Consider a classical ideal gas consisting of indistinguishable and non-interacting particles, confined in a volume V, and held at a constant temperature T. The particles are ultra-relativistic, which means that their energy E is related to their momentum $\vec{p} = (p_1, p_2, p_3)$ according to $E = |\vec{p}|c$, where c is the speed of light.

(a) Show that in the canonical ensemble, the single-particle partition function is given by

$$Z_1 = \frac{8\pi V}{(\beta ch)^3}$$

where $\beta = (k_B T)^{-1}$. **[3]**

(b) Use this result to find the *grand* canonical potential $\mathcal{Z}$ for this gas. **[3]**

(c) Using $\Phi = -k_B T \ln \mathcal{Z}$, express the chemical potential $\mu(T, P)$ as a function of the temperature T and the pressure P. **[4]**

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4. Consider a quantum system consisting of a set of single-particle microstates $|r\rangle$ of energy E_r, held at constant temperature T and chemical potential μ, and populated by indistinguishable particles. The probability $p(n_r)$ of finding occupation number n_r in the state $|r\rangle$ is given by

$$p(n_r) = \frac{e^{-\beta(E_r - \mu)n_r}}{\mathcal{Z}_r}.$$

(a) Use this expression to find $\mathcal{Z}_r$, and hence derive the average occupation number $\langle n_r \rangle$ assuming the particles are bosons. **[3]**

(b) Find $\mathcal{Z}_r$ and derive the expectation value $\langle n_r \rangle$ assuming the particles are fermions. **[3]**

(c) Sketch $\langle n_r \rangle$ as a function of E_r for bosons at some nonzero temperature. What is the range of possible values for the chemical potential μ? **[4]**

SECTION B

5. Consider a particle in one dimension whose position q and momentum p are specified by a probability distribution with the following PDF:

$$\rho(q,p) = \mathcal{N} \exp[-\lambda(p^2 + q^2)] \,, \tag{5.1}$$

where λ is a positive constant and q and p are real numbers. It may be helpful to recall the formula for a Gaussian integral, $\int_{-\infty}^{\infty} dx \exp(-ax^2) = \sqrt{\pi/a}$.

(a) Compute the coefficient $\mathcal{N}$, assuming that the PDF is properly normalized. **[2]**

(b) Compute the unconditional PDF for q, $\rho_{\text{unc}}(q)$, and the conditional PDF for q given p, $\rho_{\text{con}}(q|p)$. Comment on any similarities or differences between the two PDFs. **[4]**

(c) Using the unconditional PDF $\rho_{\text{unc}}(q)$, compute the mean $\langle q \rangle$ and the characteristic function $\tilde{\rho}_{\text{unc}}(k) = \langle e^{-ikq} \rangle$. Write down a formula for the nth moment $\langle q^n \rangle$ in terms of the characteristic function. (Hint: You may shift the integration contour by an imaginary constant: this does not introduce complications in this case.) **[6]**

(d) This PDF is an equilibrium distribution for a system with Hamiltonian $\mathcal{H} = \mathcal{H}(q,p)$. Write down one possible choice of $\mathcal{H}(q,p)$ such that this is true. Justify your answer. **[3]**

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6. Consider a 1-dimensional system with Hamiltonian:

$$\mathcal{H}(q,p) = \frac{p^4}{4m} + \frac{1}{2} m\omega^2 q^2 \,, \tag{6.1}$$

where $m > 0$ and $\omega > 0$ are positive constants and p, q are real numbers.

Recall that the Poisson bracket is defined as $\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$, and Liouville's equation is given by $\frac{\partial \rho}{\partial t} + \{\rho, \mathcal{H}\} = 0$. You may find helpful the integral

$$\int_{-\infty}^{\infty} dx \exp(-x^4) = 2\Gamma(5/4) \,.$$

(a) Write down Hamilton's equations for this system. Compute the total time derivative of the Hamiltonian, $\frac{d\mathcal{H}}{dt}$. **[3]**

(b) Compute the Poisson brackets $\{\mathcal{H}, p^n\}$ and $\{\mathcal{H}, pq\}$. **[3]**

(c) Consider an ensemble with

$$\rho(t = 0) = \frac{m\omega}{\Gamma(\frac{5}{4})\sqrt{2\pi}} \exp(-(p^4 + 2m^2\omega^2 q^2)) \,. \tag{6.2}$$

What is $\rho(t)$ for $t > 0$? **[3]**

(d) Let $\mathcal{O}(p,q,t) = tp^4$. Compute $\frac{d}{dt}\langle \mathcal{O} \rangle$ for the ensemble in (6.2). **[6]**

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7. A simple model for a double-stranded molecule such as DNA consists of a zipper with N links. Each link has a state in which it is closed (with energy zero), and a state in which it is open (with energy $\varepsilon > 0$). The zipper can only unzip from one end, say the left, and a link may only open if all the links to the left of it are already open.

(a) Show that the canonical partition function for this system takes the form

$$Z = \frac{1 - e^{-a\beta\varepsilon}}{1 - e^{-c\beta\varepsilon}}$$

where $\beta = (k_B T)^{-1}$, and a and c are constants that you should find. **[4]**

(b) Find the average number of open links $\langle n \rangle$. **[4]**

(c) What is the leading behaviour of $\langle n \rangle$ in the limit of low temperatures $k_B T \ll \varepsilon$? Does this depend on the number of links N? **[3]**

(d) What fraction of the links are open in the limit of high temperatures $k_B T \gg \varepsilon$? **[4]**

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8. The partition function for an ideal classical gas of N distinguishable particles is

$$Z_N = \left(\frac{V}{\lambda^3}\right)^N, \qquad \lambda = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2}$$

where m is the mass of the particles, V the volume and T the temperature.

(a) Use Z_N to find the entropy S of this gas. **[4]**

(b) What would the partition function and entropy be if the particles were instead *indistinguishable*?

You may use Stirling's approximation $\ln N! \approx N \ln N - N$. **[3]**

(c) State what it means for the entropy $S(T, V, N)$ to be *extensive*. In which case (distinguishable or indistinguishable particles) is the entropy extensive, and why? **[3]**

(d) Consider now a mixing process in which two initially separated ideal gases, of particle numbers N_1 and N_2 and volumes V_1 and V_2 respectively, are allowed to mix at constant temperature and pressure. After mixing, the final particle number is $N_1 + N_2$ and the final volume is $V_1 + V_2$. Both gases consist of particles of the same mass m.

Evaluate the change in entropy for this process, assuming the entropy of both gases is extensive. **[5]**
